

SHAPE DIFFERENTIABILITY OF DRAG FUNCTIONAL AND BOUNDARY VALUE PROBLEM SOLUTIONS FOR FLUID MIXTURE EQUATIONS

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Abstract: Problems of optimal design of various elements of technical structures stimulate mathematical statements of new problems of continuum mechanics and hydrodynamics in particular. This study refers to problems of shape optimization of profiles in a flow of fluid or gas. The paper deals with properties of solutions and their functional of inhomogeneous boundary value problem for nonlinear composite type partial differential equation system, simulating the a mixture of viscous compressible fluids (gases) flowing around an obstacle. Methods of the theory of partial differential equations, functional analysis and, in particular, the results on the solvability of boundary value problems for transport and Stokes equations established the well-posedness of a linear boundary value problem with singular coefficients (the problem of the original problem solution difference). This result allowed to obtain the uniqueness theorem to determine the nature of solutions dependence on the shape of the flow range and to prove domain differentiability of the solutions considered. Domain differentiability of the solution functional reflecting the force of the obstacle resistance to the incident flow is proved. A formula to equate this derivative as a sum of two summands, one of which clearly depends on mapping setting the domain shape, and the other can be expressed in terms of the so-called adjoint state, depending only on the solution of the original problem in a non-deformed domain. The functional derivative formulas may be used as the basis for building a numerical algorithm for finding the optimal shape of the body in a flow of mixture of viscous compressible fluids.

Keywords: boundary value problem, mixture of viscous compressible fluids, conjugate problem, flowing around an obstacle, material derivative, shape derivative

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INTRODUCTION

Search for the optimal shape (with the lowest drag) of the obstacle in a flow of a mixture of viscous compressible fluids (gases) is eventually associated with the problem of derivation of the shape functional, expressing the force of the obstacle resistance to the incident flow. This problem, in turn, requires a study of well-posedness of inhomogeneous boundary value problem for the corresponding equations and study of dependence of this boundary value problem solutions on the flow region shape.

Most of the known results for Navier-Stokes equations for viscous compressible fluids and moreover for equations of mixtures of such media concerns flows in are as bounded by impenetrable walls, while the results of study of inhomogeneous boundary value problems remain fairly modest. Among the papers dealing with the last issue, we'd like to mention [1], which proves existence theorem for non-stationary Navier-Stokes equations for viscous compressible fluid with constant boundary value conditions, and [2], which establishes existence of a weak solution of barotropic viscous gas flow equations in

convex domains with the outlet, independent on the time variable. Local strong solutions (close to a uniform flow) of stationary problems with inhomogeneous boundary value conditions were studied in [3–5] for two-dimensional domains on the hypothesis that the velocity field at the boundary of the flow region is close to a prescribed constant. Important results relating to the existence of strong solutions of inhomogeneous boundary value problems for stationary Navier-Stokes equations in case of small Reynolds and Mach numbers were obtained in [6–8]. Results on the well-posedness of an inhomogeneous boundary value problem for the equations of mixtures of viscous compressible fluids were obtained in [9].

The shape optimization theory is a section of variational calculus, where the functional arguments are shapes of geometrical and physical objects. A classic example of the shape optimization problem is the isoparametric Newton's problem of the body of least resistance. Description of the general theory and bibliography on this subject are available in [10–15]. The first global result concerning dependence on the solution region of compressible Navier-Stokes equations was obtained by Feireisl [16], and was further developed

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in a series of papers of Plotnikov, Ruban, Sokolowski [7,8,17–20]. These studies also provide an algorithm for calculation of the drag functional derivatives determined in the collection of domains.

The problem statement is as follows. Range of flow of viscous compressible fluid mixture is a domain $\Omega = B \setminus S$ of Euclidean space \mathbb{R}^3 of points $x = (x_1, x_2, x_3)$, external with respect to an obstacle S (which is assumed to be a compact set) and bounded by a closed surface Σ . Let us assume that $x \mapsto \vec{T}(x)$

$$\sum_{j=1}^2 L_{ij}(\vec{u}_\varepsilon^{(j)}) + \text{Re} \rho_{i\varepsilon} (\vec{u}_\varepsilon^{(i)} \cdot \nabla) \vec{u}_\varepsilon^{(i)} + \frac{\text{Re}}{\text{Ma}^2} \nabla p_{i\varepsilon}(\rho_{i\varepsilon}) + \vec{J}^{(i)} = 0 \text{ in } \Omega_\varepsilon, i=1,2, \quad (1)$$

$$\text{div}(\rho_{i\varepsilon} \vec{u}_\varepsilon^{(i)}) = 0 \text{ in } \Omega_\varepsilon, i=1,2, \quad (2)$$

where $\vec{u}_\varepsilon^{(1)}, \vec{u}_\varepsilon^{(2)}$ represent the velocity fields of the mixture components; $\rho_{1\varepsilon}, \rho_{2\varepsilon}$ are the component density functions, and corresponding pressures $p_{i\varepsilon} = p_{i\varepsilon}(\rho_{i\varepsilon}), i=1,2$, are assumed to be sufficiently smooth functions of their densities; Re and Ma denote the

$$\begin{aligned} \mu_{11} > 0, 4\mu_{11} \cdot \mu_{22} - (\mu_{12} + \mu_{21})^2 > 0, \lambda_{11} + 2\mu_{11} > 0, \\ 4(\lambda_{11} + 2\mu_{11})(\lambda_{22} + 2\mu_{22}) - (\lambda_{12} + 2\mu_{12} + \lambda_{21} + 2\mu_{21})^2 > 0. \end{aligned}$$

Summands $\vec{J}^{(i)} = (-1)^{(i)} a(\vec{u}_\varepsilon^{(2)} - \vec{u}_\varepsilon^{(1)})$, $a = \text{const} > 0, i=1,2$, describe intensity of the momentum exchange between the mixture components [22,23]. Equations (1) and (2) represent the laws of conservation of momentum and mass of the mixture components, respectively.

denotes the vector field of class $C^2(\mathbb{R}^3)$, equal to zero in the neighborhood of the boundary Σ . Let us define a mapping $x \mapsto y = \vec{T}_\varepsilon(x) = x + \varepsilon \vec{T}(x)$ which defines the perturbation of the obstacle S . For small ε , mapping $x \mapsto \vec{T}_\varepsilon(x)$ is a diffeomorphism of the flow region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$, where $S_\varepsilon = \vec{T}_\varepsilon(S)$ is perturbed obstacle in a flow.

Stationary motion of the mixture of viscous compressible fluids in the region Ω_ε is described by the following equations [21]:

Reynolds and Mach numbers, respectively; $L_{ij}, i, j=1,2$, refers to the second-order differential operators $L_{ij}(\vec{u}^{(j)}) = -\mu_{ij} \Delta \vec{u}^{(j)} - (\mu_{ij} + \lambda_{ij}) \nabla \text{div} \vec{u}^{(j)}, i, j=1,2$, where constant (dimensionless) viscosity coefficients μ_{ij}, λ_{ij} satisfy the conditions

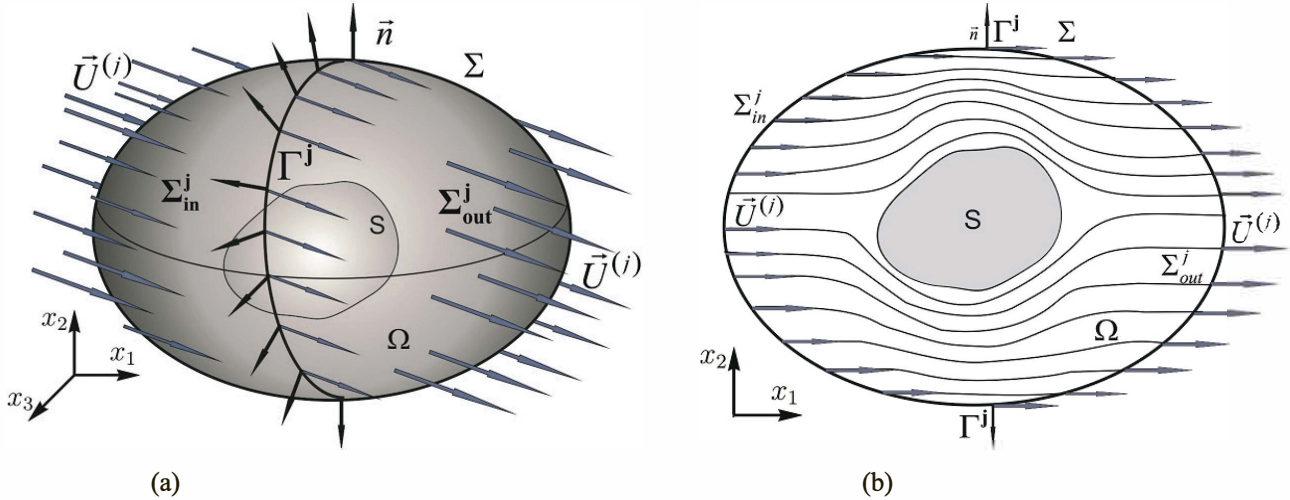


Fig. 1. Diagrams of the flow of the j th component of the mixture around the obstacle: (a) three-dimensional flow; (b) plane section.

For statement of boundary value conditions let us use the vector fields $\vec{U}^{(j)}, j=1,2$, of class $C^3(\mathbb{R}^3)$, vanishing in a neighborhood of the set S . Let us use vector functions $\vec{U}^{(j)}$ on the boundary Σ of the domain B to allocate “inflow” areas: $\Sigma_{in}^j = \{x \in \Sigma : \vec{U}^{(j)} \cdot \vec{n} < 0\}, j=1,2$, and “outflow” areas: $\Sigma_{out}^j = \{x \in \Sigma : \vec{U}^{(j)} \cdot \vec{n} > 0\}, j=1,2$ (see Fig. 1). Let us assume that the following conditions are satisfied.

Condition 1. Sets $\Gamma^j = \partial \Sigma_{in}^j \cap (\Sigma \setminus \Sigma_{in}^j), j=1,2$, (“characteristic” surface areas) are closed one-dimensional varifolds, such that $\Sigma = \Sigma_{in}^j \cup \Gamma^j \cup \Sigma_{out}^j$, and, among other things: $\int_{\Sigma} \vec{U}^{(j)} \cdot \vec{n} \, ds = 0, j=1,2; \vec{U}^{(j)} \cdot \nabla(\vec{U}^{(j)} \cdot \vec{n}) > C > 0$ on $\Gamma^j, j=1,2$, where $C > 0$ is a constant.

Adjoin the following boundary value conditions to equations (1), (2)

$$\vec{u}_\varepsilon^{(j)} = \vec{U}^{(j)} \text{ on } \Sigma, \quad \vec{u}_\varepsilon^{(j)} = 0 \text{ on } \partial S_\varepsilon, \quad \rho_{j\varepsilon} = \rho_{j\varepsilon}^0 \text{ on } \Sigma_{in}^j, j=1,2, \quad (3)$$

where $\rho_{j\varepsilon}^0, j=1,2$, are prescribed positive constants.

The force of drag to incident flow from the obstacle S_ε is expressed by the formula

$$J_D(S_\varepsilon) = -\vec{U}^\infty \sum_{i=1}^2 \int_{\partial S_\varepsilon} \left(\sum_{j=1}^2 \left[\mu_{ij} \left(\nabla \vec{u}_\varepsilon^{(j)} + \left(\nabla \vec{u}_\varepsilon^{(j)} \right)^T \right) + \lambda_{ij} \operatorname{div} \vec{u}_\varepsilon^{(j)} \mathbf{I} \right] - \frac{Re}{Ma^2} p_i(\rho_\varepsilon) \mathbf{I} \right) \cdot \vec{n} ds, \quad (4)$$

where \vec{U}^∞ is the constant vector that simulates the flow rate at “infinity”. The problem of minimization this functional is solution to the problem of selection the optimal shape of the obstacle.

The problem (1)-(3) can be conveniently reduced to

$$\vec{u}^{(i)}(x) = N(x) \vec{u}_\varepsilon^{(i)}(x + \varepsilon \vec{T}(x)), \rho_i(x) = \rho_{i\varepsilon}(x + \varepsilon \vec{T}(x)), x \in \Omega, i = 1, 2,$$

where $N(x) = (\det M(x)) M^{-1}(x)$, $M(x) = I + \varepsilon D\vec{T}(x)$, $D\vec{T}(x) = \left\{ \frac{\partial T_i(x)}{\partial x_j} \right\}$ is the Jacobi matrix of the mapping $x \mapsto \vec{T}(x)$.

As a result of this transformation, the problem (1)-(3) is transformed into the problem

$$\begin{aligned} \sum_{j=1}^2 \mu_{ij} \Delta \vec{u}^{(j)} - \nabla q_i &= \sum_{j=1}^2 \mu_{ij} \mathcal{A}(\vec{u}^{(j)}; N) + \operatorname{Re} \mathcal{B}(\rho_i, \vec{u}^{(i)}, \vec{u}^{(i)}; N) + (-1)^i \mathcal{S}(\vec{u}^{(2)} - \vec{u}^{(1)}; N) \text{ in } \Omega, \\ \operatorname{div} \vec{u}^{(i)} &= \sum_{j=1}^2 g \sigma_{ij} p_j - \sum_{j=1}^2 g \gamma_{ij} q_j \text{ in } \Omega, \\ \vec{u}^{(i)} \cdot \nabla \rho_i + \rho_i \sum_{j=1}^2 g \sigma_{ij} p_j &= \rho_i \sum_{j=1}^2 g \gamma_{ij} q_j \text{ in } \Omega, \\ \vec{u}^{(i)} &= \vec{U}^{(i)} \text{ on } \Sigma, \vec{u}^{(i)} = 0 \text{ on } \partial S, \rho_i = \rho_i^0 \text{ on } \Sigma_{in}^i, i = 1, 2. \end{aligned} \quad (5)$$

Where $g = g(x; N) = \sqrt{\det N(x)}$; linear operators \mathcal{A} , \mathcal{S} and non-linear mapping \mathcal{B} are defined according to the formulas

$$\begin{aligned} \mathcal{A}(\vec{u}; N) &= \Delta \vec{u} - (N^T)^{-1} \operatorname{div} (g^{-1} N N^T \nabla (N^{-1} \vec{u})), \\ \mathcal{B}(\rho, \vec{u}, \vec{w}; N) &= \rho (N^T)^{-1} (\vec{u} \cdot \nabla (N^{-1} \vec{w})), \quad \mathcal{S}(\vec{u}; N) = g \cdot a (N^T)^{-1} N^{-1} \vec{u}; \end{aligned}$$

$q_i = -\sum_{j=1}^2 g^{-1} (\mu_{ij} + \lambda_{ij}) \operatorname{div} \vec{u}^{(j)} + \frac{Re}{Ma^2} p_i(\rho_i)$, $i = 1, 2$, are effective viscous pressures; γ_{ij} are elements of the matrix \varkappa^{-1} ,

inverse of the matrix \varkappa which elements are $\mu_{ij} + \lambda_{ij}$, $i, j = 1, 2$; $\sigma_{ij} = \frac{Re}{Ma^2} \gamma_{ij}$.

The solution to (5) is built in the form of disturbance of specially selected sufficiently smooth flow $\vec{u}_*^{(i)}, \rho_i^*, q_i^*$, i. e.

$$\vec{u}^{(i)} = \vec{u}_*^{(i)} + \vec{v}^{(i)}, \rho_i = \rho_i^* + \varphi_i, q_i = q_i^* + \pi_i + \Lambda \cdot p_i(\rho_i^*) + \sum_{j=1}^2 (\mu_{ij} + \lambda_{ij}) m_j,$$

where $\rho_i^* = \rho_i^0 = \text{const}$, $\Lambda = \frac{Re}{Ma^2}$, m_j are constants serving for control the mass of the mixture components in a domain Ω . As a result, a problem for perturbations [9] becomes the main object of study:

$$\begin{aligned} \sum_{j=1}^2 \mu_{ij} \Delta \vec{v}^{(j)} - \nabla \pi_i &= \sum_{j=1}^2 \mu_{ij} \mathcal{A}(\vec{v}^{(j)}; N) + \operatorname{Re} \mathcal{B}(\rho_i, \vec{u}^{(i)}, \vec{v}^{(i)}; N) + (-1)^i \mathcal{S}(\vec{v}^{(2)} - \vec{v}^{(1)}; N) \text{ in } \Omega, \\ \operatorname{div} \vec{v}^{(i)} &= g \sum_{j=1}^2 \frac{1}{\rho_i} \tau_{ij} \varphi_j - g \Phi_i[\vec{\theta}] - g m_i \text{ in } \Omega, \\ \vec{v}^{(i)} \cdot \nabla \varphi_i + \tau_{ii} \varphi_i &= \Psi_i[\vec{\theta}] + g m_i \rho_i \text{ in } \Omega, \\ \vec{v}^{(i)} &= 0 \text{ on } \partial \Omega, \quad \varphi_i = 0 \text{ on } \Sigma_{in}^i, \quad \Pi \pi_i = \pi_i \left(\Pi q = q - \frac{1}{|\Omega|} \int_{\Omega} q dx \right), i = 1, 2; \\ \vec{m} &= (m_1, m_2)^T, \vec{m} = (kI - A)^{-1} \vec{f}, A = \{a_{ij}\}_{i,j=1}^2, \vec{f} = (f_1, f_2)^T, \\ k &= \int_{\Omega} g dx, a_{ij} = \frac{1}{\rho_i} \int_{\Omega} g \cdot \rho_j \cdot \zeta_i^{(j)} dx, f_i = \frac{1}{\rho_i} \int_{\Omega} \left(\sum_{j=1}^2 \zeta_i^{(j)} \cdot \Psi_j[\vec{\theta}] - g \Phi_i[\vec{\theta}] \right) dx; \\ &-\operatorname{div}(\vec{u}^{(i)} \cdot \zeta_j^{(i)}) + \tau_{ii} \cdot \zeta_j^{(i)} = \tau_{ji} g \text{ in } \Omega, \zeta_j^{(i)} = 0 \text{ on } \Sigma_{out}^i, i, j = 1, 2. \end{aligned} \quad (6)$$

Here, the constant parameters τ_{ij} are related in a known manner with the viscosity coefficients

μ_{ij}, λ_{ij} and prescribed boundary values ρ_i^0 ; $\Phi_i[\vec{\theta}], \Psi_i[\vec{\theta}]$ are the known functions of the vector $\vec{\theta} = (\vec{v}^{(1)}, \vec{v}^{(2)}; \pi_1, \pi_2, \varphi_1, \varphi_2)$ components, expressions of which are given in [9], where the existence theorem is proved as well.

Theorem 1. Let the surface Σ and vector fields $\vec{U}^{(1)}, \vec{U}^{(2)}$ satisfy the condition 1. Then there are such numbers $\sigma^* > 1$ and $\tau^* \in (0, 1)$ that if the matrix \mathbf{N} is selected based on the condition $\|I - \mathbf{N}\|_{W^2(\Omega)} \leq \tau^2, \tau \in (0, \tau^*], p_j(\rho_j) \in C^3(0, \infty), j = 1, 2,$

and the problem parameters are such that $\frac{1}{\Lambda} \leq \tau^2,$

$\text{Re} \leq \tau^2, a \leq \tau^2, |\tau_{12}| \leq \tau, |\tau_{21}| \leq \tau, \tau_{ii} \geq \sigma^*, i = 1, 2, \tau \in (0, \tau^*],$ then the problem (6) has a solution $\vec{\theta} = (\vec{v}^{(1)}, \vec{v}^{(2)}; \pi_1, \pi_2, \varphi_1, \varphi_2), m_i, \zeta_i^{(j)}, i, j = 1, 2,$

$$\|u\|_{W^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega}, |u|_{s,r,\Omega} = \int_{\Omega \times \Omega} |x-y|^{-3} \left(\frac{|u(x)-u(y)|}{|x-y|^s} \right)^r dx dy.$$

In general, the space $W^{l+s,r}(\Omega), 0 < s < 1, 1 \leq r < \infty, l \geq 0$ is an integer and is defined as the space of measurable functions with a finite norm.

$$\|u\|_{W^{l+s,r}(\Omega)} = \|u\|_{W^{l,r}(\Omega)} + \sup_{|\alpha|=l} \|D^\alpha u\|_{W^{s,r}(\Omega)}.$$

$W_0^{s,r}(\Omega), 0 \leq s \leq 1$ refers to a closed subspace $W^{s,r}(\mathbb{R}^3)$ consisting of all functions $u \in W^{s,r}(\mathbb{R}^3)$ vanishing outside the domain Ω . For the purpose of $0 < s < 1, 1 < r < \infty,$ let $\mathcal{W}_0^{s,r}(\Omega)$ denote the interpolation space $[W_0^{0,r}(\Omega), W_0^{1,r}(\Omega)]_{s,r}$ with the norm determined by the real interpolation.

Let us assume that $0 \leq s \leq 1, 1 < r < \infty, \frac{1}{r} + \frac{1}{r'} = 1.$ Let

$\mathcal{W}^{-s,r}(\Omega)$ denote completion of $L^r(\Omega)$ in the norm

$$\|v\|_{\mathcal{W}^{-s,r}(\Omega)} = \sup_{u \in \mathcal{W}_0^{s,r'}(\Omega)} \left| \int_{\Omega} u \cdot v dx \right|$$

$$\|u\|_{\mathcal{W}_0^{s,r'}(\Omega)} = 1$$

$$\mathcal{U}^{s,r} = \mathcal{W}^{s-1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}, \mathcal{V}^{s,r} = W^{s+1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R},$$

$$\mathcal{Z}^{s,r} = \mathcal{W}^{s-1,r}(\Omega) \cap L^2(\Omega)$$

$$\mathcal{E}^{s,r} = \mathcal{Z}^{s,r} \times X^{s,r} \times \mathbb{R}, \mathcal{F}^{s,r} = V^{s,r} \times X^{s,r} \times \mathbb{R},$$

Membership of the vector magnitude \vec{F} of the direct product of spaces $W_1 \times W_2 \times W_3$ is to be understood in the sense that \vec{F} is made up of three components (scalar or vector), separated by semicolon $\vec{F} = (F_1; F_2; F_3)$ and at the same time $F_1 \in W_1, F_2 \in W_2, F_3 \in W_3.$ If $W_1 = W_2 = W_3 = W,$ write $\vec{F} \in W$ and separate the vector components by comma.

DOMAIN DEPENDENCE OF SOLUTIONS

The most important stage of the shape optimization study is to prove the uniqueness of solution of the

$$\vec{q}_i = (\vec{v}_i^{(1)}, \vec{v}_i^{(2)}; \pi_1^i, \pi_2^i, \varphi_1^i, \varphi_2^i, \zeta_{1i}^{(1)}, \zeta_{2i}^{(1)}, \zeta_{1i}^{(2)}, \zeta_{2i}^{(2)}; m_1^i, m_2^i), i = 0, 1,$$

such that $\vec{\theta} \in V^{s,r} \times X^{s,r}, m_i \in \mathbb{R}, \zeta_i^{(j)} \in X^{s,r},$ where parameters $s \in (0, 1), r \in (1, \infty)$ satisfy the conditions

$s \cdot r > 3, 2s - \frac{3}{r} < 1.$ Thus vector $\vec{\theta}$ belongs to a ball B_τ of radius τ centered at the zero point of the space $V^{s,r} \times X^{s,r}.$

Here, $V^{s,r}, X^{s,r}$ refer to spaces

$$X^{s,r} = W^{s,r}(\Omega) \cap W^{1,2}(\Omega), V^{s,r} = W^{s+1,r}(\Omega) \cap W^{2,2}(\Omega)$$

where $W^{l,p}(\Omega)$ (l is a non-negative integer, $1 \leq p < \infty$) is a standard space by S.L. Sobolev [24], consisting of measurable in Ω functions having generalized derivatives in Ω of up to l order inclusively, summed with an index of power $p.$ For real $s \in (0, 1), r \in (1, \infty)$ a functional space $W^{s,r}(\Omega)$ is obtained by the real interpolation method [25] between $L^r(\Omega)$ and $W^{1,r}(\Omega),$ and consists of measurable functions with a finite norm.

As is known from [7], space $\mathcal{W}^{-s,r}(\Omega)$ for $0 < s < 1, 1 < r < \infty$ is topologically and algebraically isomorphic to Banach space $(\mathcal{W}_0^{s,r'}(\Omega))'$, dual to $\mathcal{W}_0^{s,r'}(\Omega),$ and can be identified with it.

For any $0 \leq s \leq 1, 1 < r < \infty,$ Banach space $\mathbb{W}^{-s,r}(\Omega)$ is completion of $L^r(\Omega)$ in the norm.

$$\|v\|_{\mathbb{W}^{-s,r}(\Omega)} = \|\mathcal{L}_v\|_{(\mathcal{W}^{s,r'}(\Omega))'},$$

where $\mathcal{L}_v(u) = \langle v, u \rangle = \int_{\Omega} v(x) \cdot u(x) dx$ is a continuous functional in the space $W^{s,r'}(\Omega),$ continuously embedded in $L^r(\Omega).$ As is known [7] that if Ω is a bounded domain in \mathbb{R}^3 with a boundary of class $C^1,$ then $\mathbb{W}^{-s,r}(\Omega)$ is algebraically and topologically isomorphic to the dual space $(W^{s,r'}(\Omega))',$ and can be identified with it. In addition, let us introduce functional spaces

problem (6) and differentiability of its solution with reference to the parameter $\varepsilon.$ For this purpose it is necessary to study dependence of these solutions on the matrix \mathbf{N} completely determined by the flow range deformation.

At this stage, the matrix \mathbf{N} structure is irrelevant and therefore the results obtained below are valid for an arbitrary smooth matrix-valued function $\mathbf{N}(x), x \in \Omega.$

1. The problem for differences. For difference $\vec{q}_0 - \vec{q}_1,$

of solutions to the problem (6) obtained according to Theorem 1 and corresponding to different matrices N_0 and N_1 , let us introduce the following notations

$$\bar{w}^{(j)} = \bar{v}_0^{(j)} - \bar{v}_1^{(j)}, \omega_j = \pi_j^0 - \pi_j^1, \psi_j = \varphi_j^0 - \varphi_j^1, n_j = m_j^0 - m_j^1, \xi_j^{(k)} = \zeta_{j0}^{(k)} - \zeta_{j1}^{(k)}, k, j = 1, 2. \quad (7)$$

From (6), it follows that the vector function

$$\bar{q}_0 - \bar{q}_1 = (\bar{w}^{(1)}, \bar{w}^{(2)}; \omega_1, \omega_2, \psi_1, \psi_2, \xi_1^{(1)}, \xi_2^{(1)}, \xi_1^{(2)}, \xi_2^{(2)}; n_1, n_2)$$

is the solution to the following linear problem

$$\sum_{j=1}^2 \mu_{ij} \Delta \bar{w}^{(j)} - \nabla \omega_i = \sum_{j=1}^2 \mu_{ij} \mathcal{A}_0(\bar{w}^{(j)}) + \text{Re} \mathcal{L}(\psi_i, \bar{w}^{(i)}) + \mathcal{D}_i + (-1)^i (\mathcal{E} + \mathcal{S}_0(\bar{w}^{(2)} - \bar{w}^{(1)})) \quad (8)$$

$$\text{div} \bar{w}^{(i)} = \sum_{j=1}^2 \alpha_{ij} \psi_j + \sum_{j=1}^2 \beta_{ij} \omega_j + \gamma_i n_i + \delta_i d, \quad (9)$$

$$\bar{u}_0^{(i)} \cdot \nabla \psi_i + \tau_i \psi_i = -\bar{w}^{(i)} \cdot \nabla \varphi_i^1 + \sum_{j=1}^2 \hat{\alpha}_{ij} \psi_j + \sum_{j=1}^2 \hat{\beta}_{ij} \omega_j + \hat{\gamma}_i n_i + \hat{\delta}_i d, \quad (10)$$

$$- \text{div}(\bar{u}_0^{(i)} \xi_j^{(i)}) + \tau_i \xi_j^{(i)} = \text{div}(\bar{w}^{(i)} \zeta_{j1}^{(i)}) + \tau_{ij} d \text{ in } \Omega, \quad (11)$$

$$\bar{w}^{(i)} = 0 \text{ on } \Omega, \psi_i = 0 \text{ on } \Sigma_n^i, \omega_i = \Pi \omega_i, \xi_j^{(i)} = 0 \text{ on } \Sigma_{out}^i, \quad (12)$$

$$n_i = \sum_{k=1}^2 \chi_{ik}^0 \int_{\Omega} \tilde{\delta}_k d + \sum_{j=1}^2 \left(\tilde{\alpha}_{kj} \psi_j + \tilde{\beta}_{kj} \omega_j + \tilde{\gamma}_{kj} \xi_j^{(k)} \right) dx, i, j = 1, 2. \quad (13)$$

These equations use the following notations

$$\mathcal{A}_k(\bar{w}) = \mathcal{A}(\bar{w}; N_k), \mathcal{B}_k(\rho, \bar{u}, \bar{w}) = \mathcal{B}(\rho, \bar{u}, \bar{w}; N_k), \mathcal{S}_k(\bar{w}) = \mathcal{S}(\bar{w}; N_k), k = 0, 1,$$

$$\mathcal{L}(\psi_i, \bar{w}^{(i)}) = \mathcal{B}_0(\psi_i, \bar{u}_0^{(i)}, \bar{u}_0^{(i)}) + \mathcal{B}_0(\rho_i^1, \bar{w}^{(i)}, \bar{u}_0^{(i)}) + \mathcal{B}_0(\rho_i^1, \bar{u}_1^{(i)}, \bar{w}^{(i)}),$$

$$\mathcal{E} = \mathcal{S}_0(\bar{u}_1^{(2)} - \bar{u}_1^{(1)}) - \mathcal{S}_1(\bar{u}_1^{(2)} - \bar{u}_1^{(1)}),$$

$$\mathcal{D}_i = \sum_{j=1}^2 \mu_{ij} \left(\mathcal{A}_0(\bar{u}_1^{(j)}) - \mathcal{A}_1(\bar{u}_1^{(j)}) \right) + \text{Re} \left(\mathcal{B}_0(\rho_i^1, \bar{u}_1^{(i)}, \bar{u}_1^{(i)}) - \mathcal{B}_1(\rho_i^1, \bar{u}_1^{(i)}, \bar{u}_1^{(i)}) \right),$$

$$d = g_0 - g_1, g_k = \sqrt{\det N_k}.$$

Symbols χ_{ik}^0 refer to elements of the matrix $(k(N_0)I - A(N_0, \bar{\theta}_0))^{-1}$. Coefficients $\alpha_{ij}, \beta_{ij}, \gamma_i, \delta_i, \hat{\alpha}_{ij}, \hat{\beta}_{ij}, \hat{\gamma}_i, \hat{\delta}_i, \tilde{\alpha}_{ij}, \tilde{\beta}_{ij}, \tilde{\gamma}_{ij}, \tilde{\delta}_i, i, j = 1, 2$, in the right-hand members of equations (9), (10) and (13) depend

on solutions \bar{q}_0 and \bar{q}_1 and are considered as known functions. Expressions for these coefficients are very bulky, whereby they are not stated here. Let us point out only their estimates that will be required in the future. Under the conditions of Theorem 1, the inequalities

$$\begin{cases} \|\alpha_{ij}\|_{X^{s,r}}, i \neq j, \|\beta_j\|_{X^{s,r}}, \|\delta_i\|_{X^{s,r}}, \|\hat{\alpha}_{ij}\|_{X^{s,r}}, \|\hat{\beta}_{ij}\|_{X^{s,r}}, \|\hat{\delta}_i\|_{X^{s,r}} \leq c\tau, \\ \|\tilde{\alpha}_{ij}\|_{X^{s,r}}, \|\tilde{\beta}_j\|_{X^{s,r}}, \|\tilde{\gamma}_{ij}\|_{X^{s,r}}, \|\tilde{\delta}_i\|_{X^{s,r}} \leq c\tau, \\ \|\alpha_{ii}\|_{X^{s,r}}, \|\gamma_i\|_{X^{s,r}}, \|\hat{\gamma}_i\|_{X^{s,r}} \leq c, i, j = 1, 2, \end{cases} \quad (14)$$

are valid. Parameter τ indicates the radius of a ball in the space $V^{s,r} \times X^{s,r}$, to which solutions $\bar{\theta}_0$ and $\bar{\theta}_1$ belong. c denotes a constant depending only on the domain Ω , vector fields $\bar{U}^{(1)}, \bar{U}^{(2)}$ and parameters $r, s, \tau_{11}, \tau_{22}$. Subsequently dependence of a particular constant on $\Omega, \bar{U}^{(1)}, \bar{U}^{(2)}, \tau_1, \tau_2$ will be referred to as the dependence on the problem data.

2. Conjugate problem. The feature of the problem (8)-(13) for the difference $\bar{q}_0 - \bar{q}_1$ is that the known results [7] on transport equations are not applicable to equations (10), because the summand $\bar{w}^{(i)} \cdot \nabla \varphi_i^1$

does not satisfy the required smoothness conditions (is neither smooth nor bounded). However, the existence theorem implies that summands $\bar{w}^{(i)} \cdot \nabla \varphi_i^1$ belong to space $\mathcal{W}^{s-1,r}(\Omega)$, dual to $\mathcal{W}_0^{1-s,r'}(\Omega)$, and therefore it is possible to treat the difference $\bar{q}_0 - \bar{q}_1$ as a very weak solution to the problem (8)-(13). In this regard, let us formulate the conjugate problem as follows.

For given vector fields $\bar{H}^{(i)}$, scalar fields G_i, F_i, M_{1i}, M_{2i} and constants $s_i, i = 1, 2$, it is necessary to find the vector fields $\bar{w}_*^{(i)}$, scalar fields $\omega_i^*, \psi_i^*, \xi_i^{(j)*}$ and constants $n_i^*, i, j = 1, 2$, such that

$$\sum_{j=1}^2 \mu_{ij} \left(\Delta \bar{w}_*^{(j)} - \nabla \omega_j^* - \mathcal{A}_0(\bar{w}_*^{(j)}) \right) - \text{Re} \mathcal{H}_i(\bar{w}_*^{(i)}) + (-1)^{i+1} \mathcal{S}_0(\bar{w}_*^{(2)} - \bar{w}_*^{(1)}) + \psi_i^* \cdot \nabla \varphi_i^1 + \sum_{j=1}^2 \zeta_{j1}^{(i)} \cdot \nabla \xi_j^{(i)*} = \bar{H}^{(i)} \text{ in } \Omega, \quad (15)$$

$$div \vec{w}_*^{(i)} = \Pi \sum_{k=1}^2 \left[\hat{\beta}_{ki} \psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\beta}_{ji} + \mu_{kj} \beta_{ji} \omega_k^*) \right] + \Pi G_i \text{ in } \Omega, \quad (16)$$

$$-div(\psi_i^* \vec{u}_0^{(i)}) + \tau_{ii} \psi_i^* = Re \mathcal{M}_i(\vec{w}_*^{(i)}) + \sum_{k=1}^2 \left[\hat{\alpha}_{ki} \psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\alpha}_{ji} + \mu_{kj} \alpha_{ji} \omega_k^*) \right] + F_i \text{ in } \Omega, \quad (17)$$

$$\vec{u}_0^{(i)} \nabla \xi_j^{(i)*} + \tau_{ii} \xi_j^{(i)*} = \tilde{\gamma}_{ij} \sum_{k=1}^2 n_k^* \chi_{ki}^0 + M_{ji} \text{ in } \Omega, \quad (18)$$

$$n_i^* = \int_{\Omega} \left(\gamma_i \sum_{j=1}^2 \mu_{ji} \omega_j^* + \hat{\gamma}_i \psi_i^* \right) dx + s_i, \quad (19)$$

$$\vec{w}_*^{(i)} = 0 \text{ on } \partial\Omega, \psi_i^* = 0 \text{ on } \Sigma_{out}^i, \xi_j^{(i)*} = 0 \text{ on } \Sigma_{in}^i, \Pi \omega_i^* = \omega_i^*, i, j = 1, 2. \quad (20)$$

Linear operators \mathcal{H}_i and \mathcal{M}_i are defined by the following formulas

$$\mathcal{H}_i(\vec{h}) = \rho_i^1 \nabla (\mathbf{N}_0^{-1} \vec{u}_0^{(i)}) \mathbf{N}_0^{-1} \vec{h} - (\mathbf{N}_0^T)^{-1} div(\rho_i^1 \vec{u}_1^{(i)} \otimes (\mathbf{N}_0^{-1} \vec{h})),$$

$$\mathcal{M}_i(\vec{h}) = (\vec{u}_0^{(i)} \nabla (\mathbf{N}_0^{-1} \vec{u}_0^{(i)}) \cdot \mathbf{N}_0^{-1} \vec{h}, i = 1, 2.$$

The further content of this section is devoted to proving existence and uniqueness of strong and weak solutions of the conjugate problem (15)-(20). These results enable us to derive estimates of norms for the differences $\vec{w}^{(i)}$, ω_i , ψ_i , $\xi_i^{(j)}$, n_i .

Theorem 2. *Let the conditions of the existence theorem are satisfied and parameters s, r are such that $\frac{1}{2} < s < 1, (1-s)r > 3$. Then there are numbers c, σ_c and τ_c depending on the objective and parameters s, r such that if $\min\{\tau_1, \tau_2\} > \sigma_c$ and $0 < \tau < \tau_c$, then for every vector function $\vec{f} = (\vec{f}^{(1)}, \vec{f}^{(2)})$, $\vec{f}^{(i)} = (\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i) \in \mathcal{U}^{s,r}$, the problem (15)-(20) has a unique solution $\vec{h} = (\vec{h}^{(1)}, \vec{h}^{(2)})$,*

$\vec{h}^{(i)} = (\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_1^{(i)*}, \xi_2^{(i)*}; n_i^*) \in \mathcal{V}^{s,r}$ satisfying the inequality

$$\|\vec{h}\|_{\mathcal{V}^{s,r}} \leq c \|\vec{f}\|_{\mathcal{U}^{s,r}}. \quad (21)$$

If we apply a more restrictive condition to the right-hand member \vec{f} , namely $\vec{f} \in \mathcal{E}^{s,r}$, the solution \vec{h} belongs to the class $\mathcal{F}^{s,r}$ and at the same time the following inequality is valid.

$$\|\vec{h}\|_{\mathcal{F}^{s,r}} \leq c \|\vec{f}\|_{\mathcal{E}^{s,r}}. \quad (22)$$

Theorem 2 proof scheme

Equations (15)-(19) of the conjugate problem can be represented in a symbolic form

$$A[\vec{h}_*] - B[\vec{h}_*] = \vec{F}, \vec{h}_* = (\vec{h}_*^{(1)}, \vec{h}_*^{(2)}) \vec{h}_*^{(i)} = (\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_1^{(i)*}, \xi_2^{(i)*}; n_i^*)$$

where the integral-differential operators A and B are determined by the formulas

$$A[\vec{h}_*] = (A_1[\vec{h}_*], A_2[\vec{h}_*]),$$

$$A_i[\vec{h}_*] = \left\{ \begin{array}{l} \sum_{j=1}^2 \mu_{ji} (\Delta \vec{w}_*^{(j)} - \nabla \omega_j^*) + \sum_{j=1}^2 \zeta_{j1}^{(i)} \cdot \nabla \xi_j^{(i)*} \\ div \vec{w}_*^{(i)} \\ -\vec{u}_0^{(i)} \nabla \psi_i^* + \tau_{ii} \psi_i^* - \sum_{k=1}^2 \sum_{j=1}^2 \mu_{kj} \alpha_{ji} \omega_k^* \\ \vec{u}_0^{(i)} \nabla \xi_1^{(i)*} + \tau_{ii} \xi_1^{(i)*} \\ \vec{u}_0^{(i)} \nabla \xi_2^{(i)*} + \tau_{ii} \xi_2^{(i)*} \\ n_i^* - \int_{\Omega} \left(\gamma_i \sum_{j=1}^2 \mu_{ji} \omega_j^* + \hat{\gamma}_i \psi_i^* \right) dx \end{array} \right\}$$

$$B[\vec{h}_*] = (B_1[\vec{h}_*], B_2[\vec{h}_*]), \quad (23)$$

$$B_i[\vec{h}_*] = \left\{ \begin{array}{l} \sum_{j=1}^2 \mu_{ji} A_0(\vec{w}_*^{(j)}) + \text{Re} \mathcal{H}_i(\vec{w}_*^{(i)}) + (-1)^i S_0(\vec{w}_*^{(2)} - \vec{w}_*^{(1)}) - \psi_i^* \cdot \nabla \phi_i^1 \\ \Pi \sum_{k=1}^2 \left[\hat{\beta}_{ki} \psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\beta}_{ji} + \mu_{kj} \beta_{ji} \omega_k^*) \right] \\ \text{Re} \mathcal{M}_i(\vec{w}_*^{(i)}) + \psi_i^* \text{div} \vec{u}_0^{(i)} + \sum_{k=1}^2 \left[\hat{\alpha}_{ki} \psi_k^* + n_k^* \chi_{kj}^0 \tilde{\alpha}_{ji} \right] \\ \tilde{\gamma}_{i1} \sum_{k=1}^2 n_k^* \chi_{ki}^0 \\ \tilde{\gamma}_{i2} \sum_{k=1}^2 n_k^* \chi_{ki}^0 \\ 0 \end{array} \right.$$

Let us consider the following boundary value problem by connecting to the system of equations

$$A[\vec{h}_*] = \vec{F} \tag{24}$$

the boundary conditions

$$\vec{w}_*^{(j)} = 0 \text{ on } \partial\Omega, \psi_i^* = 0 \text{ on } \Sigma_{out}^i, \xi_1^{(i)*} = 0, \xi_2^{(i)*} = 0 \text{ on } \Sigma_{in}^i, \Pi \omega_i^* = \omega_i^*, i = 1, 2. \tag{25}$$

It is easy to see that for each right-hand member $\vec{F} = (\vec{F}^{(1)}, \vec{F}^{(2)})$, $\vec{F}^{(i)} = (\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i)$ of the space $\mathcal{U}^{s,r}$, the boundary value problem (24), (25) is divided into several independent linear boundary value problems, namely, functions $\xi_j^{(i)*}, i, j = 1, 2$, are defined independently as solutions to the following problem for transport equation

$$\vec{u}_0^{(i)} \nabla \xi_j^{(i)*} + \tau_{ij} \xi_j^{(i)*} = M_{ji} \text{ in } \Omega, \xi_j^{(i)*} = 0 \text{ on } \Sigma_{in}^i, i, j = 1, 2.$$

$$-\vec{u}_0^{(i)} \nabla \psi_i^* + \tau_{ii} \psi_i^* = \sum_{k=1}^2 \sum_{j=1}^2 \mu_{kj} \alpha_{ji} \omega_k^* + F_i \text{ in } \Omega, \psi_i^* = 0 \text{ on } \Sigma_{out}^i, i = 1, 2.$$

The procedure for building solution to the problem (24), (25) ends with finding the constants n_i^* according to the formulas

$$n_i^* = \int_{\Omega} \left(\gamma_i \sum_{j=1}^2 \mu_{ji} \omega_j^* + \hat{\gamma}_i \psi_i^* \right) dx + s_i, i = 1, 2.$$

On the basis of known results on transport equations and linear Stokes-type problem [7,26,9] we can assert the existence of the constants c and σ_c , depending on the problem data $\Omega, \vec{U}^{(1)}, \vec{U}^{(2)}$ and parameters r, s , such that if $\min\{\tau_1, \tau_2\} \geq \sigma_c$, then the problem (24), (25) is uniquely solvable in spaces $\mathcal{V}^{s,r}$ and $\mathcal{F}^{s,r}$ and the following inequality is valid

$$\|\vec{h}_*\|_{\mathcal{V}^{s,r}} \leq c \|\vec{F}\|_{\mathcal{U}^{s,r}}, \|\vec{h}_*\|_{\mathcal{F}^{s,r}} \leq c \|\vec{F}\|_{\mathcal{E}^{s,r}}. \tag{26}$$

The foregoing can be interpreted in the form of existence of (inverse) bounded linear operator $A^{-1} : \mathcal{U}^{s,r} \rightarrow \mathcal{V}^{s,r}$ ($A^{-1} : \mathcal{E}^{s,r} \rightarrow \mathcal{F}^{s,r}$) which assigns a unique solution $\vec{h}_* = A^{-1}[\vec{F}]$ to the boundary value problem (24), (25) to the element $\vec{F} \in \mathcal{U}^{s,r}$ ($\vec{F} \in \mathcal{E}^{s,r}$).

Let us now turn to the integral-differential operator B given by the formulas (23), and note the following properties.

Lemma 1. *Let us assume that conditions of Theorem 1 and $\frac{1}{2} < s < 1, (1-s)r > 3$ are satisfied. Then B is a bounded linear operator from $\mathcal{V}^{s,r}$ to $\mathcal{U}^{s,r}$ and from $\mathcal{F}^{s,r}$ to $\mathcal{E}^{s,r}$, and the following inequalities are valid*

After this, the components $\vec{w}_*^{(i)}, \omega_i^*, i = 1, 2$, are found as solutions to the Stokes-type problems, i.e.

$$\sum_{j=1}^2 \mu_{ji} (\Delta \vec{w}_*^{(j)} - \nabla \omega_j^*) = \vec{H}^{(i)} - \sum_{j=1}^2 \zeta_{ji}^{(i)} \cdot \nabla \xi_j^{(i)*} \text{ in } \Omega,$$

$$\text{div} \vec{w}_*^{(i)} = \Pi G_i \text{ in } \Omega,$$

$$\vec{w}_*^{(i)} = 0 \text{ on } \partial\Omega, \Pi \omega_i^* = \omega_i^*, i = 1, 2.$$

The next step is definition of components ψ_i^* as the solution to the boundary value problem

$$\|B\vec{h}\|_{\mathcal{U}^{s,r}} \leq c \cdot \tau \|\vec{h}\|_{\mathcal{V}^{s,r}}, \tag{27}$$

$$\|B\vec{h}\|_{\mathcal{E}^{s,r}} \leq c \cdot \tau \|\vec{h}\|_{\mathcal{F}^{s,r}}. \tag{28}$$

Here, the constant c depends only on Ω , vector fields $\vec{U}^{(1)}, \vec{U}^{(2)}$ and parameters r, s .

Now we can complete the proof of existence and uniqueness of the conjugate problem solution.

Let us assume that conditions of Theorem 2 are satisfied. The conjugate problem (15)-(20) can be written as the operator equation

$$(I - A^{-1}B)\vec{h} = A^{-1}\vec{f}$$

Due to (26), (27) linear operators $A^{-1} : \mathcal{U}^{s,r} \rightarrow \mathcal{V}^{s,r}$ and $A^{-1}B : \mathcal{V}^{s,r} \rightarrow \mathcal{V}^{s,r}$ are bounded, moreover

$$\|A^{-1}B\|_{L(\mathcal{V}^{s,r})} \leq c_V \cdot \tau, \tag{29}$$

where the constant c_V depends only on the problem data and parameters r, s . On the basis of the conditions in Theorem 1, parameter τ satisfies the conditions $0 < \tau \leq \tau^*(\tau_1, \tau_2)$. Let us choose the constant τ_c such that the inequalities are valid

$$0 < \tau_c \leq \tau^*(\tau_1, \tau_2) \quad \tau_c \cdot c_V \leq q < 1$$

From (29) it follows that the norm $\|A^{-1}B\|_{L(\mathcal{V}^{s,r})}$ for $\tau \in (0; \tau_c]$ is separated from the unity, and on the basis of the well-known theorem, the operator $(I - A^{-1}B)^{-1}$ exists and is bounded

$$\|(I - A^{-1}B)^{-1}\|_{L(\mathcal{V}^{s,r})} \leq \frac{1}{1-q}.$$

So for these values τ and any $\vec{f} \in \mathcal{U}^{s,r}$ the conjugate problem (15)-(20) has a unique solution $\vec{h} \in \mathcal{V}^{s,r}$ and the inequality (21) is valid.

Suppose now that the right-hand member \vec{f} of the problem (15)-(20) belongs to space $\mathcal{E}^{s,r}$. As the operator A^{-1} of the solution to problem (24), (25) is a bounded linear operator from $\mathcal{E}^{s,r}$ to $\mathcal{F}^{s,r}$, then due to estimates (26), (28), we have

$$\|A^{-1}B\|_{L(\mathcal{F}^{s,r})} \leq c_F \cdot \tau.$$

If the parameter τ is subordinate to the condition $\tau \in (0; \tau_c]$, where

$$0 < \tau_c \leq \tau^*(\tau_{11}, \tau_{22}), \quad \tau_c \cdot c_F \leq q < 1,$$

$$\begin{aligned} & \sum_{i=1}^2 \left[\langle \vec{H}^{(i)}, \vec{w}^{(i)} \rangle_1 + \langle \omega_i, G_i \rangle_0 + \langle \psi_i, F_i \rangle_0 + \langle \xi_{1i}^{(i)}, M_{1i} \rangle_0 + \langle \xi_{2i}^{(i)}, M_{2i} \rangle_0 + n_i s_i \right] = \\ & = \sum_{i=1}^2 \int_{\Omega} \left[\vec{w}_*^{(i)} (\mathcal{D}_i + (-1)^i \mathcal{E}) + d \left(\delta_i \psi_i^* + \sum_{j=1}^2 (\tau_{ji} \xi_j^{(i)*} + n_i^* \chi_{ij}^0 \tilde{\delta}_j + \mu_{ij} \delta_j \omega_i^*) \right) \right] dx \end{aligned} \quad (30)$$

where $\vec{f}^{(i)} = (\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i)$, $i=1,2$, are arbitrary functions of $\mathcal{W}^{s-1,r}(\Omega) \times \mathcal{W}^{s,r}(\Omega) \times \mathbb{R}$, and $(\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_{1i}^{(i)*}, \xi_{2i}^{(i)*}; n_i^*)$, $i=1,2$, is the relevant solution to the conjugate problem (15)-(20).

Proof. Since the set $L^r(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$ is contained in $\mathcal{E}^{s,r}$ and is dense in $\mathcal{U}^{s,r} = \mathcal{W}^{s-1,r}(\Omega) \times \mathcal{W}^{s,r}(\Omega) \times \mathbb{R}$, then for any element $\vec{f} = (\vec{f}^{(1)}, \vec{f}^{(2)})$, $\vec{f}^{(i)} = (\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i)$ of space $\mathcal{U}^{s,r}$, there is a sequence $\vec{f}_n = (\vec{f}_n^{(1)}, \vec{f}_n^{(2)})$, $\vec{f}_n^{(i)} = (\vec{H}_n^{(i)}; G_n, F_n, M_{1n}, M_{2n}; s_n)$, $\vec{f}_n^{(i)} \in L^r(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$ such that

$$\vec{f}_n \rightarrow \vec{f} \text{ in } \mathcal{U}^{s,r}, \quad n \rightarrow \infty. \quad (31)$$

Therefore let us begin with prove of the identity (30) for a vector function \vec{f} from a space $L^r(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$. In this case, the relevant solution $(\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_{1i}^{(i)*}, \xi_{2i}^{(i)*}, n_i^*)$, $i=1,2$, to the conjugate problem (15)-(20) belongs to the class

$$\begin{aligned} & \sum_{i=1}^2 \left[\int_{\Omega} (\vec{w}_*^{(i)} \vec{H}^{(i)} + \omega_i G_i + \psi_i F_i + \xi_{1i}^{(i)} M_{1i} + \xi_{2i}^{(i)} M_{2i}) dx + n_i s_i \right] = \\ & = \sum_{i=1}^2 \int_{\Omega} \left[\vec{w}_*^{(i)} (\mathcal{D}_i + (-1)^i \mathcal{E}) + d \left(\delta_i \psi_i^* + \sum_{j=1}^2 (\tau_{ji} \xi_j^{(i)*} + n_i^* \chi_{ij}^0 \tilde{\delta}_j + \mu_{ij} \delta_j \omega_i^*) \right) \right] dx. \end{aligned} \quad (32)$$

Let us note that vector functions $(\vec{H}^{(i)}; \omega_i, \psi_i, \xi_{1i}^{(i)}, \xi_{2i}^{(i)})$ and $(\vec{w}_*^{(i)}; G_i, F_i, M_{1i}, M_{2i})$ belong to dual spaces $\mathcal{W}^{s-1,r}(\Omega) \times \mathbb{W}^{-s,r}(\Omega)$ and $\mathcal{W}_0^{1-s,r'}(\Omega) \times \mathcal{W}^{s,r}(\Omega)$, respectively. Indeed, $\vec{H}^{(i)} \in \mathcal{W}^{s-1,r}(\Omega)$ because $\vec{H}^{(i)} \in L^r(\Omega)$, and there is a limited embedding $L^r(\Omega)$ in $\mathcal{W}^{s-1,r}(\Omega)$. Scalar fields $\omega_i, \psi_i, \xi_{1i}^{(i)}, \xi_{2i}^{(i)}$ are elements $\mathcal{W}^{s,r}(\Omega)$, and embeddings $\mathcal{W}^{s,r}(\Omega) \subset L^r(\Omega) \subset \mathbb{W}^{-s,r'}(\Omega)$

then, similar to the previous case we obtain that under condition $\vec{f} \in \mathcal{E}^{s,r}$ solution \vec{h} to the problem (15)-(20) belongs to the class $\mathcal{F}^{s,r}$ and the estimate (22) is valid. Theorem 2 is proved.

3. Estimates of differences. For $s \in (0,1)$, $r \in (1,\infty)$ let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_0$ denote the duality relationship between pairs of spaces $\mathcal{W}^{s-1,r}(\Omega)$, $\mathcal{W}_0^{1-s,r'}(\Omega)$ and $\mathbb{W}^{-s,r'}(\Omega)$, $\mathcal{W}^{s,r}(\Omega)$, respectively.

Lemma 2. Let us assume that conditions of Theorem 2 are satisfied. Then vector fields $\vec{w}^{(j)}$, $j=1,2$, scalar fields $\omega_j, \psi_j, \xi_j^{(i)}$, $i, j=1,2$, and constants n_j , $j=1,2$, defined according to the formulas (7) belong to spaces $\mathcal{W}_0^{1-s,r'}(\Omega)$, $\mathbb{W}^{-s,r'}(\Omega)$, \mathbb{R} and satisfy the identity

$V^{s,r} \times X^{s,r} \times \mathbb{R} = \mathcal{F}^{s,r}$ (in particular, this means that vector fields $\vec{w}_*^{(i)}$ have continuous first-order derivatives and square summable second-order derivatives, and scalar functions $\omega_i^*, \psi_i^*, \xi_{1i}^{(i)*}, \xi_{2i}^{(i)*}$ are continuous and have square summable first-order derivatives). Due to Theorem 1, we can assert that $\vec{w}^{(i)}, \vec{v}_k^{(i)}, \vec{u}_k^{(i)} \in V^{s,r}$, $\omega_i, \psi_i, \xi_j^{(i)}, \rho_i^k, \pi_i^k, \phi_i^k, \zeta_{jk}^{(i)} \in X^{s,r}$, $i, j=1,2, k=0,1$.

The continuity of embeddings $V^{s,r}$ in $C^1(\Omega)$ provides continuous differentiability in Ω vector fields $\vec{w}^{(i)}, \vec{v}_k^{(i)}, \vec{u}_k^{(i)}$ and, of course, their membership of $W^{2,2}(\Omega)$. Continuity of embedding $X^{s,r}$ in $C(\Omega)$ implies continuity of scalar functions $\omega_i, \psi_i, \xi_j^{(i)}, \rho_i^k, \pi_i^k, \phi_i^k, \zeta_{jk}^{(i)}$, belonging to the same spaces $W^{1,2}(\Omega)$. This implies that equations (8)-(11), (13) are satisfied in the strong sense. Multiplying these equations by the solution $(\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_{1i}^{(i)*}, \xi_{2i}^{(i)*}, n_i^*)$, $i=1,2$, to the conjugate problem (15)-(20), corresponding to the element $\vec{f} \in L^r(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$, and integrating by parts, we obtain the identity

$(r > 3, 1 < r' = \frac{r}{r-1} < r)$ imply that these functions belong to $\mathbb{W}^{-s,r'}(\Omega)$.

Further, the vector field $\vec{w}^{(i)} \in V^{s,r}$ and hence $\vec{w}^{(i)} \in C^1(\Omega)$. In addition $\vec{w}^{(i)}$ vanishes in $\partial\Omega$. Thus, it is clear that $\vec{w}^{(i)} \in W_0^{1,r'}(\Omega)$. Since $\mathcal{W}_0^{1-s,r'}(\Omega)$ is the interpolation space $[L^r(\Omega), W_0^{1,r'}(\Omega)]_{1-s,r'}$ and $W_0^{1,r'}(\Omega) \subset L^r(\Omega)$, then $W_0^{1,r'}(\Omega) \subset \mathcal{W}_0^{1-s,r'}(\Omega)$. So,

we have $\vec{w}^{(i)} \in \mathcal{W}_0^{1-s,r'}(\Omega)$. On the other hand elements G_i, F_i, M_{1i}, M_{2i} belong to $W^{s,r}(\Omega)$. In view of the above, integrals in the left-hand member of the equality (32) can be written in terms of duality forms and thus, the

$$\begin{aligned} & \sum_{i=1}^2 \left\{ \langle \vec{H}_n^{(i)}, \vec{w}^{(i)} \rangle_1 + \langle \omega_i, G_{in} \rangle_0 + \langle \psi_i, F_{in} \rangle_0 + \langle \xi_1^{(i)}, M_{1in} \rangle_0 + \langle \xi_2^{(i)}, M_{2in} \rangle_0 + n_i s_{in} \right\} = \\ & = \sum_{i=1}^2 \int_{\Omega} \left\{ \vec{w}_n^{(i)*} (\mathcal{D}_i + (-1)^i \mathcal{E}) + d \left(\hat{\delta}_i \psi_{in}^* + \sum_{j=1}^2 (\tau_{ji} \xi_{jn}^{(i)*} + n_{in}^* \chi_{ij}^0 \tilde{\delta}_j + \mu_{ij} \delta_j \omega_{in}^*) \right) \right\} dx, \end{aligned} \tag{33}$$

where $\vec{h}_n^{(i)} = (\vec{w}_n^{(i)*}, \omega_n^*, \psi_n^*, \xi_{1n}^{(i)*}, \xi_{2n}^{(i)*}; n_n^*)$, $i=1,2$, is the solution to the problem (15)-(20), corresponding to the “right-hand member” $\vec{f}_n^{(i)}$, $i=1,2$. From (31), due to Theorem 2, we have that

$$\vec{h}_n = (\vec{h}_n^{(1)}, \vec{h}_n^{(2)}) \rightarrow \vec{h} = (\vec{h}^{(1)}, \vec{h}^{(2)}) \text{ in } \mathcal{V}^{s,r}, n \rightarrow \infty. \tag{34}$$

Passing to the limit as $n \rightarrow \infty$ in (33), due to (31),

$$\begin{aligned} & \sum_{i=1}^2 \left(\|\vec{w}^{(i)}\|_{\mathcal{W}_0^{1-s,r'}} + \|\psi_i\|_{\mathcal{W}^{-s,r'}} + \|\omega_i\|_{\mathcal{W}^{-s,r'}} + \|\xi_1^{(i)}\|_{\mathcal{W}^{-s,r'}} + \|\xi_2^{(i)}\|_{\mathcal{W}^{-s,r'}} + |n_i| \right) \leq \\ & \leq c \left(\|d\|_{L^1(\Omega)} + \|\mathcal{D}_1\|_{L^1(\Omega)} + \|\mathcal{D}_2\|_{L^1(\Omega)} + \|\mathcal{E}\|_{L^1(\Omega)} \right) \end{aligned} \tag{35}$$

where the constant c depends on the problem data and parameters r, s .

Proof. Turning to the equality (30), we note that since the coefficients $\hat{\delta}_i, \tilde{\delta}_i, \delta_i$ are bounded

$$\sum_{i=1}^2 \left(\|\vec{w}_n^{(i)*}\|_{C(\Omega)} + \|\psi_i^*\|_{C(\Omega)} + \|\omega_i^*\|_{C(\Omega)} + \|\xi_1^{(i)*}\|_{C(\Omega)} + \|\xi_2^{(i)*}\|_{C(\Omega)} + |n_i^*| \right) \left(\|d\|_{L^1(\Omega)} + \|\mathcal{D}_1\|_{L^1(\Omega)} + \|\mathcal{D}_2\|_{L^1(\Omega)} + \|\mathcal{E}\|_{L^1(\Omega)} \right) \cdot c \tag{36}$$

In addition, due to embedding theorem and estimate (21) we have

$$\sum_{i=1}^2 \left(\|\vec{w}_n^{(i)*}\|_{C(\Omega)} + \|\psi_i^*\|_{C(\Omega)} + \|\omega_i^*\|_{C(\Omega)} + \|\xi_1^{(i)*}\|_{C(\Omega)} + \|\xi_2^{(i)*}\|_{C(\Omega)} + |n_i^*| \right) \leq c \|\vec{h}\|_{\mathcal{V}^{s,r}} \leq c \|\vec{f}\|_{\mathcal{U}^{s,r}} \tag{37}$$

From (36) and (37) we obtain

$$\left| \sum_{i=1}^2 \left\{ \langle \vec{H}_n^{(i)}, \vec{w}^{(i)} \rangle_1 + \langle \omega_i, G_i \rangle_0 + \langle \psi_i, F_i \rangle_0 + \langle \xi_1^{(i)}, M_{1i} \rangle_0 + \langle \xi_2^{(i)}, M_{2i} \rangle_0 + n_i s_i \right\} \right| \leq c \sum_{i=1}^2 \left(\|d\|_{L^1(\Omega)} + \|\mathcal{D}_i\|_{L^1(\Omega)} + \|\mathcal{E}\|_{L^1(\Omega)} \right) \|\vec{f}\|_{\mathcal{U}^{s,r}} \tag{38}$$

The inequality (38) obviously implies the estimate (35). Lemma 3 is proved.

The inequality (35) implies, firstly, the uniqueness of solution to the problem (6). In addition, (35) implies that the mapping associating the matrix-valued function N with solution \vec{q} to the inhomogeneous boundary value problem (6) is a Lipschitz solution in a weak norm.

MATERIAL DERIVATIVES

So-called material derivatives of solution to the problem (6) is an important tool to

Material derivative $\vec{q}'(0) = (\vec{w}^{(1)}, \vec{w}^{(2)}; \omega_1, \omega_2, \psi_1, \psi_2, \xi_1^{(1)}, \xi_2^{(1)}, \xi_1^{(2)}, \xi_2^{(2)}; n_1, n_2)$ of , solution $\vec{q}(\varepsilon) = (\vec{v}^{(1)}(\varepsilon), \vec{v}^{(2)}(\varepsilon); \pi_1(\varepsilon), \pi_2(\varepsilon), \varphi_1(\varepsilon), \varphi_2(\varepsilon), \zeta_1^{(1)}(\varepsilon), \zeta_2^{(1)}(\varepsilon), \zeta_1^{(2)}(\varepsilon), \zeta_2^{(2)}(\varepsilon); m_1(\varepsilon), m_2(\varepsilon))$ to the problem (6) with $\varepsilon = 0$ is defined as follows:

$$\begin{aligned} \vec{w}_\varepsilon^{(j)} &= \lim_{\varepsilon \rightarrow 0} \vec{w}^{(j)}(\varepsilon), \quad \omega_j = \lim_{\varepsilon \rightarrow 0} \omega_{j\varepsilon}, \quad \psi_j = \lim_{\varepsilon \rightarrow 0} \psi_{j\varepsilon}, \quad \xi_j^{(i)} = \lim_{\varepsilon \rightarrow 0} \xi_{j\varepsilon}^{(i)}, \quad n_j = \lim_{\varepsilon \rightarrow 0} n_{j\varepsilon}, \quad i, j = 1, 2. \\ \vec{w}_\varepsilon^{(j)} &= -\varepsilon^{-1} \left(\vec{v}^{(j)}(0) - \vec{v}^{(j)}(\varepsilon) \right), \quad \omega_{j\varepsilon} = -\varepsilon^{-1} \left(\pi_j(0) - \pi_j(\varepsilon) \right), \quad \psi_{j\varepsilon} = -\varepsilon^{-1} \left(\varphi_j(0) - \varphi_j(\varepsilon) \right), \\ \xi_{j\varepsilon}^{(i)} &= -\varepsilon^{-1} \left(\zeta_j^{(i)}(0) - \zeta_j^{(i)}(\varepsilon) \right), \quad n_{j\varepsilon} = -\varepsilon^{-1} \left(m_j(0) - m_j(\varepsilon) \right), \quad i, j = 1, 2, \end{aligned} \tag{39}$$

provided that the limits in (39) exist in some sense.

identity (30) of Lemma 2 is proved for vector functions $(\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i)$ from a narrower functional space $L^r(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$, i.e, we may write

(34) we obtain the identity(30) stated in Lemma 2. Lemma 2 is proved.

Identity (30) proved in Lemma 2 means that the difference $\vec{q}_0 - \vec{q}_1$ is a very weak solution to the linear problem (8)-(13).

Lemma 3. Let the conditions of Lemma 2 are satisfied. Then the following inequality is valid

in modulus by a constant depending only on the problem data and r, s (see (14)), then the right-hand member of this identity can be estimated by a value

prove the drag functional differentiability. This section is devoted to their study. Solution $\vec{q} = (\vec{v}^{(1)}, \vec{v}^{(2)}; \pi_1, \pi_2, \varphi_1, \varphi_2, \zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_1^{(2)}, \zeta_2^{(2)}; m_1, m_2)$ to the problem (6) is a function of spatial variable $x = (x_1, x_2, x_3) \in \Omega$ and parameter ε . Let us assume that $\vec{q}(\varepsilon) = \langle \langle x \rightarrow q(x, \varepsilon) \rangle \rangle$ and consider \vec{q} (and, of course, components \vec{q}) as a function of ε with values in the space of functions of x .

Equations for material derivatives can be easily obtained by formal use of $\bar{\theta} = \bar{\theta}(\varepsilon)$ and $\mathbf{N} = \mathbf{N}(\varepsilon)$ in equations (6) and differentiating results according to

ε at zero. This formal procedure gives the following system of equations and boundary conditions for material derivatives.

$$\begin{aligned} \sum_{j=1}^2 \mu_{ij} \Delta \bar{w}^{(j)} - \nabla \omega_i &= \mathcal{L}^* (\Psi_i, \bar{w}^{(i)}) + \mathcal{D}_i^* + (-1)^i (\mathcal{E}^* + a(\bar{w}^{(2)} - \bar{w}^{(1)})) \text{ in } \Omega, \\ \operatorname{div} \bar{w}^{(i)} &= \sum_{j=1}^2 \alpha_{ij}^* \Psi_j + \sum_{j=1}^2 \beta_{ij}^* \omega_j + \gamma_i^* n_i + \delta_i^* d^* \text{ in } \Omega, \\ \bar{u}^{(i)}(0) \cdot \nabla \Psi_i + \tau_{ii} \Psi_i &= -\bar{w}^{(i)} \cdot \nabla \varphi_i(0) + \sum_{j=1}^2 \hat{\alpha}_{ij}^* \Psi_j + \sum_{j=1}^2 \hat{\beta}_{ij}^* \omega_j + \hat{\gamma}_i^* n_i + \hat{\delta}_i^* d^* \text{ in } \Omega, \\ -\operatorname{div}(\bar{u}^{(i)}(0) \xi_j^{(i)}) + \tau_{ii} \xi_j^{(i)} &= \operatorname{div}(\bar{w}^{(i)} \zeta_j^{(i)}(0)) + \tau_{ji} d^* \text{ in } \Omega, \\ \bar{w}^{(i)} &= 0 \text{ on } \partial\Omega, \quad \Psi_i = 0 \text{ on } \Sigma_{in}^i, \quad \omega_i = \Pi \omega_i, \quad \xi_j^{(i)} = 0 \text{ on } \Sigma_{out}^i, \\ n_i &= \sum_{k=1}^2 \chi_{ik}^0 \int_{\Omega} \left[\tilde{\delta}_k^* d^* + \sum_{j=1}^2 (\tilde{\alpha}_{kj}^* \Psi_j + \tilde{\beta}_{kj}^* \omega_j + \tilde{\gamma}_{kj}^* \xi_j^{(k)}) \right] dx, \quad i, j = 1, 2. \end{aligned} \quad (40)$$

These equations use the following notations

$$\begin{aligned} \mathcal{D}_i^* &= -\operatorname{Re} \rho_i(0) (\bar{u}^{(i)}(0) \cdot \nabla (\mathbb{D} \bar{u}^{(i)}(0))) + \mathbb{D}^T \bar{u}^{(i)}(0) \cdot \nabla \bar{u}^{(i)}(0) + \sum_{j=1}^2 \mu_{ij} (\operatorname{div}(\mathbb{T} \nabla \bar{u}^{(i)}(0)) + \mathbb{D}^T \Delta \bar{u}^{(i)}(0) + \Delta (\mathbb{D} \bar{u}^{(i)}(0))), \\ \mathcal{L}^* (\Psi_i, \bar{w}^{(i)}) &= \operatorname{Re} (\Psi_i \bar{u}^{(i)}(0) \cdot \nabla \bar{u}^{(i)}(0) + \rho_i(0) \bar{w}^{(i)} \cdot \nabla \bar{u}^{(i)}(0) + \rho_i \bar{u}^{(i)}(0) \cdot \nabla \bar{w}^{(i)}); \\ \mathcal{E}^* &= a \mathbb{T} (\bar{u}^{(2)}(0) - \bar{u}^{(1)}(0)); \quad d^* = \operatorname{div} \bar{T}; \quad \mathbb{D} = \operatorname{div} \bar{T} \mathbf{I} - D \bar{T}; \quad \mathbb{T} = \operatorname{div} \bar{T} \mathbf{I} - D \bar{T} - (D \bar{T})^T. \end{aligned} \quad (41)$$

Coefficients $\alpha_{ij}^*, \beta_{ij}^*, \gamma_i^*, \delta_i^*, \hat{\alpha}_{ij}^*, \hat{\beta}_{ij}^*, \hat{\gamma}_i^*, \hat{\delta}_i^*, \tilde{\alpha}_{ij}^*, \tilde{\beta}_{ij}^*, \tilde{\gamma}_{ij}^*, \tilde{\delta}_i^*, i, j = 1, 2$, in the right-hand members of equations (40) depend on solution to the problem (6), corresponding to $\varepsilon = 0$.

To prove existence of a very weak solution of the problem (40)-(41) let us introduce the conjugate problem in the following formulation: for given vector fields $\bar{H}^{(i)}$, scalar fields G_i, F_i, M_{1i}, M_{2i} and constants $s_i, i = 1, 2$, it is necessary to find vector fields $\bar{w}_*^{(i)}$, scalar fields $\omega_i^*, \Psi_i^*, \xi_j^{(j)*}$ constants $n_i^*, i, j = 1, 2$, such that

$$\begin{aligned} \sum_{j=1}^2 \mu_{ji} (\Delta \bar{w}_*^{(j)} - \nabla \omega_j^*) - \operatorname{Re} H_i^* (\bar{w}_*^{(j)}) + (-1)^{i+1} a (\bar{w}_*^{(2)} - \bar{w}_*^{(1)}) + \Psi_i^* \cdot \nabla \varphi_i(0) + \sum_{j=1}^2 \zeta_j^{(i)}(0) \nabla \xi_j^{(i)*} &= \bar{H}^{(i)} \text{ in } \Omega, \\ \operatorname{div} \bar{w}_*^{(i)} &= \Pi \sum_{k=1}^2 \left[\hat{\beta}_{ki}^* \Psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\beta}_{ji}^* + \mu_{kj} \beta_{ji}^* \omega_k^*) \right] + \Pi G_i \text{ in } \Omega, \\ -\operatorname{div}(\Psi_i^* \bar{u}^{(i)}(0)) + \tau_{ii} \Psi_i^* &= \operatorname{Re} M_i^* (\bar{w}_*^{(j)}) + \sum_{k=1}^2 \left[\hat{\alpha}_{ki}^* \Psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\alpha}_{ji}^* + \mu_{kj} \alpha_{ji}^* \omega_k^*) \right] + F_i \text{ in } \Omega, \\ \bar{u}^{(i)}(0) \nabla \xi_j^{(i)*} + \tau_{ii} \xi_j^{(i)*} &= \tilde{\gamma}_{ij}^* \sum_{k=1}^2 n_k^* \chi_{ki}^0 + M_{ji} \text{ in } \Omega, \\ n_i^* &= \int_{\Omega} \left(\gamma_i^* \sum_{j=1}^2 \mu_{ji} \omega_j^* + \hat{\gamma}_i^* \Psi_i^* \right) dx + s_i, \quad i, j = 1, 2, \\ \bar{w}_*^{(j)} &= 0 \text{ on } \partial\Omega, \quad \Psi_i^* = 0 \text{ on } \Sigma_{out}^i, \quad \xi_j^{(i)*} = 0 \text{ on } \Sigma_{in}^i, \quad \Pi \omega_i^* = \omega_i^*, \quad i, j = 1, 2. \end{aligned} \quad (42)$$

Linear operators \mathcal{H}_i^* and \mathcal{M}_i^* are defined by the following formulas

$$\mathcal{H}_i^* (\bar{h}) = \rho_i(0) \nabla (\bar{u}^{(i)}(0)) \bar{h} - \operatorname{div}(\rho_i(0) \bar{u}^{(i)}(0) \otimes \bar{h}), \quad \mathcal{M}_i^* (\bar{h}) = (\bar{u}^{(i)}(0) \nabla \bar{u}^{(i)}(0)) \cdot \bar{h}, \quad i = 1, 2. \quad (43)$$

Well-posedness of the conjugate problem (42) statement is the result of the following lemma.

Lemma 4. *Let us assume that conditions of Theorem 2 are satisfied. Then for each vector function $\bar{f} = (\bar{f}^{(1)}, \bar{f}^{(2)})$, $\bar{f}^{(i)} = (\bar{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i) \in \mathcal{U}^{s,r}$, the problem (42) has a unique solution $\bar{h}_* = (\bar{h}_*^{(1)}, \bar{h}_*^{(2)})$, $\bar{h}_*^{(i)} = (\bar{w}_*^{(i)}; \omega_i^*, \Psi_i^*, \xi_1^{(i)*}, \xi_2^{(i)*}; n_i^*) \in \mathcal{V}^{s,r}$.*

Proof. Problem (42) is a special case of the problem (15)-(20), with $N_0 = \mathbf{I}$ and $\mathbf{g}_0 = 1$, $\bar{q}_i = \bar{q}(0)$ $\bar{u}_i^{(j)} = \bar{u}_*^{(j)} + \bar{v}^{(j)}(0)$,

$\rho_j^i = \rho_j^* + \varphi_j(0)$, $\zeta_{ji}^{(k)} = \zeta_j^{(k)}(0)$ $i = 0,1; k, j = 1,2$.
Therefore, the lemma is a direct consequence of Theorem 2.

Linear equations (40) are obtained by formal differentiation, and the fact that the limits $\vec{w}^{(i)}, \omega_i, \psi_i, \xi_j^{(i)}, n_i, i, j = 1,2$ in (39) really exist and satisfy (40), is not obvious. Further it will be shown that they exist in a weak sense, are defined as the elements of the space $\mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$ and defined as a very weak solution of the linear problem (40).

The proof of these facts is preceded by a brief comment of a technical nature.

Lemma 5. *There are numbers $\varepsilon_1 > 0$ and c, c_1 , depending on $\|\vec{T}\|_{C^2(\Omega)}$ such that for any $\varepsilon \in [0; \varepsilon_1]$, the following representations exist*

$$\mathbf{N}(\varepsilon)^{\pm 1} = \mathbf{I} \pm \varepsilon \mathbb{D} + \varepsilon^2 \mathbb{D}^{\pm}, \mathbf{g}(\varepsilon)^{\pm 1} = 1 \pm \varepsilon \text{div} \vec{T} + \varepsilon^2 \mathbf{g}^{\pm} \quad (44)$$

$$\begin{aligned} & \sum_{i=1}^2 \left\{ \langle \vec{H}^{(i)}, \vec{w}^{(i)} \rangle_1 + \langle \omega_i, G_i \rangle_0 + \langle \psi_i, F_i \rangle_0 + \langle \xi_{1i}^{(i)}, M_{1i} \rangle_0 + \langle \xi_{2i}^{(i)}, M_{2i} \rangle_0 + n_i s_i \right\} = \\ & = \sum_{i=1}^2 \int_{\Omega} \left\{ \vec{w}_*^{(i)} (\mathcal{D}_i^* + (-1)^i \mathcal{E}^*) + d^* \left(\hat{\delta}_i^* \psi_i^* + \sum_{j=1}^2 (\tau_{ji} \xi_j^{(i)*} + n_i^* \chi_{ij} \tilde{\delta}_j^* + \mu_{ij} \delta_j^* \omega_i^*) \right) \right\} dx, \end{aligned} \quad (45)$$

Here $\vec{h}_* = (\vec{h}_*^{(1)}, \vec{h}_*^{(2)})$, $\vec{h}_*^{(i)} = (\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_{1i}^{(i)*}, \xi_{2i}^{(i)*}; n_i^*) \in \mathcal{V}^{s,r}$ is the solution to problem (42) defined in Lemma 4; \mathcal{D}_i^* , \mathcal{E}^* and d^* are defined by the formulas (41).

We confine ourselves to a short scheme of proof of Theorem 3. Establishing the boundedness of sequence of functions $\vec{h}_\varepsilon = (\vec{h}_\varepsilon^{(1)}, \vec{h}_\varepsilon^{(2)})$, $\vec{h}_\varepsilon^{(i)} = (\vec{w}_\varepsilon^{(i)}; \omega_{i\varepsilon}, \psi_{i\varepsilon}, \xi_{i\varepsilon}^{(1)}, \xi_{i\varepsilon}^{(2)}; n_{i\varepsilon})$ in reflexive space $\mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$ and passing if necessary to a subsequence, we obtain existence of vector functions $(\vec{w}^{(i)}; \omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)}; n_i) \in \mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$

$$\begin{aligned} & \sum_{i=1}^2 \left\{ \langle \vec{H}^{(i)}, \vec{w}_\varepsilon^{(i)} \rangle_1 + \langle \omega_{i\varepsilon}, G_i \rangle_0 + \langle \psi_{i\varepsilon}, F_i \rangle_0 + \langle \xi_{1\varepsilon}^{(i)}, M_{1i} \rangle_0 + \langle \xi_{2\varepsilon}^{(i)}, M_{2i} \rangle_0 + n_{i\varepsilon} s_i \right\} = \\ & = -\frac{1}{\varepsilon} \sum_{i=1}^2 \int_{\Omega} \left\{ \vec{w}_*^{(i)} \cdot (\mathcal{D}_i(\varepsilon) + (-1)^i \mathcal{E}(\varepsilon)) + d(\varepsilon) \left(\hat{\delta}_i(\varepsilon) \psi_{i\varepsilon}^* + \sum_{j=1}^2 (\tau_{ji} \xi_j^{(i)*} + n_{i\varepsilon}^* \chi_{ij} \tilde{\delta}_j(\varepsilon) + \mu_{ij} \delta_j(\varepsilon) \omega_{i\varepsilon}^*) \right) \right\} dx, \end{aligned} \quad (46)$$

where $\vec{h}_{*\varepsilon}^{(i)} = (\vec{w}_{*\varepsilon}^{(i)}; \omega_{i\varepsilon}^*, \psi_{i\varepsilon}^*, \xi_{1i\varepsilon}^{(i)*}, \xi_{2i\varepsilon}^{(i)*}; n_{i\varepsilon}^*)$, $i = 1,2$ is the solution to problem the conjugate to the problem for the function $\vec{h}_\varepsilon = (\vec{h}_\varepsilon^{(1)}, \vec{h}_\varepsilon^{(2)})$, $\vec{h}_\varepsilon^{(i)} = (\vec{w}_\varepsilon^{(i)}; \omega_{i\varepsilon}, \psi_{i\varepsilon}, \xi_{i\varepsilon}^{(1)}, \xi_{i\varepsilon}^{(2)}; n_{i\varepsilon})$. Since space

$$\begin{aligned} & \sum_{i=1}^2 \left\{ \langle \vec{H}^{(i)}, \vec{w}_\varepsilon^{(i)} \rangle_1 + \langle \omega_{i\varepsilon}, G_i \rangle_0 + \langle \psi_{i\varepsilon}, F_i \rangle_0 + \langle \xi_{1\varepsilon}^{(i)}, M_{1i} \rangle_0 + \langle \xi_{2\varepsilon}^{(i)}, M_{2i} \rangle_0 + n_{i\varepsilon} s_i \right\} \rightarrow \\ & \rightarrow \sum_{i=1}^2 \left\{ \langle \vec{H}^{(i)}, \vec{w}^{(i)} \rangle_1 + \langle \omega_i, G_i \rangle_0 + \langle \psi_i, F_i \rangle_0 + \langle \xi_{1i}^{(i)}, M_{1i} \rangle_0 + \langle \xi_{2i}^{(i)}, M_{2i} \rangle_0 + n_i s_i \right\} \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (47)$$

Further, by setting a weak convergence of sequence of solutions $\vec{h}_{*\varepsilon} = (\vec{h}_{*\varepsilon}^{(1)}, \vec{h}_{*\varepsilon}^{(2)})$, $\vec{h}_{*\varepsilon}^{(i)} = (\vec{w}_{*\varepsilon}^{(i)}; \omega_{i\varepsilon}^*, \psi_{i\varepsilon}^*, \xi_{1i\varepsilon}^{(i)*}, \xi_{2i\varepsilon}^{(i)*}; n_{i\varepsilon}^*)$, to the conjugate

where $\mathbb{D} = \text{div} \vec{T} \mathbf{I} - D \vec{T}$ and $\|\mathbb{D}^{\pm}\|_{C^2(\Omega)} + \|\mathbf{g}^{\pm}\| \leq c$, and deviations $\mathbf{N}(\varepsilon)^{\pm 1} - \mathbf{I}$, $\mathbf{g}(\varepsilon)^{\pm 1} - 1$ satisfy the inequality

$$\|\mathbf{N}(\varepsilon)^{\pm 1} - \mathbf{I}\|_{C^2(\Omega)} + \|\mathbf{g}(\varepsilon)^{\pm 1} - 1\|_{C^2(\Omega)} \leq c_1 \varepsilon$$

The proof of Lemma 5 is, for example, in [7].

Theorem 3. *Let the conditions of Theorem 2 and $\varepsilon \in [0; \varepsilon_*]$ are satisfied, where $\varepsilon_* = \min\{c_1^{-1} \tau^2, \varepsilon_1\}$ and the constants c_1 and ε_1 are chosen in accordance with Lemma 5. Then there are such $\vec{h} = (\vec{h}^{(1)}, \vec{h}^{(2)})$ $\vec{h}^{(i)} = (\vec{w}^{(i)}; \omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)}; n_i) \in \mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$ that $\vec{w}_\varepsilon^{(i)} \rightarrow \vec{w}^{(i)}$ weakly in $\mathcal{W}_0^{1-s,r'}(\Omega)$, $i = 1,2$, $(\omega_{i\varepsilon}, \psi_{i\varepsilon}, \xi_{i\varepsilon}^{(1)}, \xi_{i\varepsilon}^{(2)}) \rightarrow (\omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)})$ weakly in $\mathbb{W}^{-s,r'}(\Omega)$, $i = 1,2$, $n_{i\varepsilon} \rightarrow n_i \in \mathbb{R}$ at $\varepsilon \rightarrow 0$. In addition, for any $(\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; e_i) \in \mathcal{U}^{s,r}$, $i = 1,2$, the identity is valid*

such that $\vec{w}_\varepsilon^{(i)} \rightarrow \vec{w}^{(i)}$ weakly in $\mathcal{W}_0^{1-s,r'}(\Omega)$, $(\omega_{i\varepsilon}, \psi_{i\varepsilon}, \xi_{i\varepsilon}^{(1)}, \xi_{i\varepsilon}^{(2)}) \rightarrow (\omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)})$ weakly in $\mathbb{W}^{-s,r'}(\Omega)$, $n_{i\varepsilon} \rightarrow n_i \in \mathbb{R}$ as $\varepsilon \rightarrow 0$. On the other hand, for arbitrary $\vec{f} = (\vec{f}^{(1)}, \vec{f}^{(2)})$, $\vec{f}^{(i)} = (\vec{H}^{(i)}; G_i, F_i, M_{1i}, M_{2i}; s_i) \in \mathcal{W}^{s-1,r}(\Omega) \times \mathcal{W}^{s,r}(\Omega) \times \mathbb{R}$ be valid

$\mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$ is dual to space $\mathcal{W}^{s-1,r}(\Omega) \times \mathcal{W}^{s,r}(\Omega) \times \mathbb{R}$, then passing to the limit as $\varepsilon \rightarrow 0$, we obtain

problem in space $\mathcal{W}^{s+1,r}(\Omega) \times \mathcal{W}^{s,r}(\Omega) \times \mathbb{R}$ to the solution $\vec{h}_* = (\vec{h}_*^{(1)}, \vec{h}_*^{(2)})$, $\vec{h}_*^{(i)} = (\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_{1i}^{(i)*}, \xi_{2i}^{(i)*}; n_i^*)$, of the problem (42) and considering the compactness of

embedding $W^{s+1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}$ in $C(\Omega) \times \mathbb{R}$ we obtain that $\vec{h}_{\varepsilon} \rightarrow \vec{h}$ in $C(\Omega) \times \mathbb{R}, i=1,2$.

Properties of coefficients of equations for difference

$$-\frac{1}{\varepsilon} \sum_{i=1}^2 \left\{ \int_{\vec{w}_{\varepsilon}^{(i)}} \left(\mathcal{D}_i(\varepsilon) + (-1)^i \mathcal{E}(\varepsilon) \right) + d(\varepsilon) \left(\hat{\delta}_i(\varepsilon) \Psi_{i\varepsilon}^* + \sum_{j=1}^2 \left(\tau_{ji} \xi_{j\varepsilon}^{(i)} + n_{i\varepsilon}^* \chi_{ij} \tilde{\delta}_j(\varepsilon) + \mu_{ij} \delta_j(\varepsilon) \omega_{i\varepsilon}^* \right) \right) dx \right\} \rightarrow$$

$$\rightarrow \sum_{i=1}^2 \int_{\Omega} \left\{ \vec{w}_{\varepsilon}^{(i)} (\mathcal{D}_i^* + (-1)^i \mathcal{E}^*) + d^* \left(\hat{\delta}_i^* \Psi_i^* + \sum_{j=1}^2 \left(\tau_{ji} \xi_j^{(i)*} + n_i^* \chi_{ij}^0 \tilde{\delta}_j^* + \mu_{ij} \delta_j^* \omega_i^* \right) \right) \right\} dx \quad (48)$$

Equations (46)-(48) prove the identity(45).

Thus it is proved that every weak limit $\vec{h} = (\vec{h}^{(1)}, \vec{h}^{(2)})$, $\vec{h}^{(i)} = (\vec{w}^{(i)}; \omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)}; n_i)$, of the sequence $\vec{h}_{\varepsilon} = (\vec{h}_{\varepsilon}^{(1)}, \vec{h}_{\varepsilon}^{(2)})$, $\vec{h}_{\varepsilon}^{(i)} = (\vec{w}_{\varepsilon}^{(i)}; \omega_{i\varepsilon}, \psi_{i\varepsilon}, \xi_{i\varepsilon}^{(1)}, \xi_{i\varepsilon}^{(2)}; n_{i\varepsilon})$ satisfies the identity (45). It is necessary to demonstrate that the sequence has a unique weak limit point, and hence converges weakly to this point. Let us assume that there are $(\vec{w}^{(i)}; \omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)}; n_i), i=1,2$, and $(\vec{w}'^{(i)}; \omega'_i, \psi'_i, \xi_i'^{(1)}, \xi_i'^{(2)}; n'_i), i=1,2$, of $\mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$ that satisfy (45). Then the following identity is valid for all $(\vec{H}^{(i)}, G_i, F_i, M_{1i}, M_{2i}, s_i) \in \mathcal{W}^{s-1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}, i=1,2$,

$$\sum_{i=1}^2 \left\{ \langle \vec{H}^{(i)}, \vec{w}^{(i)} - \vec{w}'^{(i)} \rangle_1 + \langle \omega_i - \omega'_i, G_i \rangle_0 + \langle \psi_i - \psi'_i, F_i \rangle_0 + \langle \xi_1^{(i)} - \xi_1'^{(i)}, M_{1i} \rangle_0 + \langle \xi_2^{(i)} - \xi_2'^{(i)}, M_{2i} \rangle_0 + (n_i - n'_i) s_i \right\} = 0.$$

Since the space $\mathcal{W}_0^{s-1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}$ is dual to $\mathcal{W}_0^{1-s,r'}(\Omega) \times \mathbb{W}^{-s,r'}(\Omega) \times \mathbb{R}$, this identity implies that $(\vec{w}^{(i)}; \omega_i, \psi_i, \xi_i^{(1)}, \xi_i^{(2)}; n_i) = (\vec{w}'^{(i)}; \omega'_i, \psi'_i, \xi_i'^{(1)}, \xi_i'^{(2)}; n'_i), i=1,2$.

The following lemma is the consequence of the results obtained in the proof of Theorem 3

Lemma 6. *Let us assume that conditions of Theorem 2 are satisfied. Then $\vec{q}(\varepsilon) \rightarrow \vec{q}(0)$ in $W^{s+1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}$ as $\varepsilon \rightarrow 0$.*

DRAG FUNCTIONAL DOMAIN DERIVATIVE

Let us assume the conditions of Theorem 3 are satisfied. Then each matrix $\mathbf{N}(\varepsilon)$ corresponds to a unique solution $\vec{\theta}(\varepsilon) = (\vec{v}^{(1)}(\varepsilon), \vec{v}^{(2)}(\varepsilon), \pi_1(\varepsilon), \pi_2(\varepsilon), \varphi_1(\varepsilon), \varphi_2(\varepsilon))$ to the

problem (6), which in turn induces the solution

$$\vec{u}^{(i)}(\varepsilon) = \vec{u}_{\varepsilon}^{(i)} + \vec{v}^{(i)}(\varepsilon), \rho_i(\varepsilon) = \rho_i^* + \varphi_i(\varepsilon), q_i(\varepsilon) = q_i^* + \pi_i(\varepsilon) + \Lambda \cdot p_i(\rho_i^*) + \sum_{j=1}^2 \beta_{ij} m_j, i=1,2, \quad (49)$$

to the problem (5). Turning to the unperturbed domain Ω in the formula (4) for drag functional, we obtain the following representation.

$$J_D S_{\varepsilon} = J(\Omega_{\varepsilon}) = - \sum_{i=1}^2 U_{\infty} \cdot \left\langle \int_{\Omega} \text{Re} \eta \rho_i \vec{u}^{(i)} \nabla (\mathbf{N}^{-1} \vec{u}^{(i)}) dx + \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} g^{-1} \left(\mathbf{N}^T \nabla (\mathbf{N}^{-1} \vec{u}^{(j)}) + \nabla (\mathbf{N}^{-1} \vec{u}^{(j)})^T \mathbf{N} - \text{div}(\vec{u}^{(j)}) \mathbb{I} \right) - q_i \mathbb{I} \right) \mathbf{N}^T \nabla \eta dx \right\rangle \quad (50)$$

where η is the arbitrary smooth function identically equal to 1 in the neighborhood of the obstacle S and identically equal to zero in the neighborhood of the boundary Σ . Substituting functions (49) into the expression (50), we obtain:

$$J(\Omega_{\varepsilon}) = \sum_{i=1}^2 \left(U_{\infty} \cdot J_1^i(\Omega_{\varepsilon}) + U_{\infty} \cdot J_2^i(\Omega_{\varepsilon}) \right) \quad (51)$$

where

$$J_1^i(\Omega_{\varepsilon}) = - \int_{\Omega} \text{Re} \eta \rho_i(\varepsilon) \vec{u}^{(i)}(\varepsilon) \nabla (\mathbf{N}(\varepsilon)^{-1} \vec{u}^{(i)}(\varepsilon)) dx$$

$$J_2^i(\Omega_{\varepsilon}) = \int_{\Omega} q_i(\varepsilon) \mathbf{N}(\varepsilon)^T \nabla \eta dx - \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} g(\varepsilon)^{-1} \left(\mathbf{N}(\varepsilon)^T \nabla (\mathbf{N}(\varepsilon)^{-1} \vec{u}^{(j)}(\varepsilon)) + \nabla (\mathbf{N}(\varepsilon)^{-1} \vec{u}^{(j)}(\varepsilon))^T \mathbf{N}(\varepsilon) - \text{div}(\vec{u}^{(j)}(\varepsilon)) \mathbb{I} \right) \right) \mathbf{N}(\varepsilon)^T \nabla \eta dx$$

Calculation of the drag and its variations with respect to perturbations of S is a problem of practical importance, for example, to build a numerical algorithm for finding the optimal shape of the obstacle in the flow. In this section, an explicit formula for the following derivative is obtained

$$d(\Omega)[\vec{T}] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (J(\Omega_{\varepsilon}) - J(\Omega)) \quad (52)$$

Let us calculate $dJ(\Omega)[\vec{T}]$ in terms of material derivatives of solutions to the problem (6). The corresponding result is given in the following theorem,

which is a direct consequence of the Theorem 3.

Theorem 4. Suppose that all the conditions of Theorem 3 are satisfied. Then derivative $dJ(\Omega)[\vec{T}]$ defined by the formula (52) exists and can be represented as

$$d(\Omega)[\vec{T}] = L_u(\vec{T}) + L_e(\vec{T}). \quad (53)$$

$$\begin{aligned} B_i &= (\mu_{1i} + \mu_{2i})\Delta\eta U_\infty + Re\rho_i(0)(\vec{u}^{(i)}(0) \cdot \nabla\eta) U_\infty - Re\rho_i(0)\nabla\vec{u}^{(i)}(0) \cdot U_\infty\eta, \\ A &= \nabla\eta \cdot U_\infty, \quad C_i = -Re\eta(\vec{u}^{(i)}(0) \cdot \nabla\vec{u}^{(i)}(0)) \cdot U_\infty, \end{aligned} \quad (55)$$

and L_e takes the form

$$\begin{aligned} L_e(\vec{T}) &= -\sum_{i=1}^2 U_\infty Re \int_{\Omega} [\rho_i(0)(\vec{u}^{(i)}(0) \cdot \nabla\eta) \mathbb{D}\vec{u}^{(i)}(0)] dx + \\ &+ \sum_{i=1}^2 U_\infty \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} \left(\nabla(\vec{u}^{(j)}(0)) + \nabla(\vec{u}^{(j)}(0))^T - \text{div}(\vec{u}^{(j)}(0))\mathbf{I} \right) \right) (D\vec{T})^T \nabla\eta dx - \\ &- \sum_{i=1}^2 U_\infty \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} \left(\mathbb{D}^T \nabla(\vec{u}^{(j)}(0)) - \nabla(\mathbb{D}\vec{u}^{(j)}(0)) + (\mathbb{D}^T \nabla(\vec{u}^{(j)}(0)) - \nabla(\mathbb{D}\vec{u}^{(j)}(0)))^T \right) - q_i(0)\mathbb{D}^T \right) \nabla\eta dx, \end{aligned} \quad (56)$$

where $\mathbb{D} = \text{div}\vec{T} \mathbf{I} - D\vec{T}$.

Proof of Theorem 4. Substituting relationships

$$\vec{u}^{(i)}(\varepsilon) - \vec{u}^{(i)}(0) = \varepsilon\vec{w}_\varepsilon^{(i)}, \quad \rho_i(\varepsilon) - \rho_i(0) = \varepsilon\psi_{i\varepsilon}, \quad q_i(\varepsilon) - q_i(0) = \varepsilon\omega_{i\varepsilon} + \varepsilon \sum_{j=1}^2 \beta_{ij} n_{\varepsilon j}, \quad i = 1, 2,$$

in (51), we obtain the difference relationship

$$\frac{1}{\varepsilon} (J(\Omega_\varepsilon) - J(\Omega)) = \sum_{i=1}^2 \left(U_\infty \cdot \frac{1}{\varepsilon} (J_1^i(\Omega_\varepsilon) - J_1^i(\Omega)) + U_\infty \cdot \frac{1}{\varepsilon} (J_2^i(\Omega_\varepsilon) - J_2^i(\Omega)) \right),$$

which can be represented as:

$$\frac{1}{\varepsilon} (J(\Omega_\varepsilon) - J(\Omega)) = L_{\varepsilon,u} + L_{\varepsilon,e} \quad (57)$$

where

$$\begin{aligned} L_{\varepsilon,u} &= -\sum_{i=1}^2 Re U_\infty \int_{\Omega} \eta [\rho_i(\varepsilon)\vec{w}_\varepsilon^{(i)} \nabla\vec{u}^{(i)}(\varepsilon) + \psi_{i\varepsilon} \vec{u}^{(i)}(0) \nabla\vec{u}^{(i)}(\varepsilon) + \rho_i(0)\vec{u}^{(i)}(0) \nabla\vec{w}_\varepsilon^{(i)}] dx \\ &- \sum_{i=1}^2 U_\infty \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} \left(\nabla\vec{w}_\varepsilon^{(j)} + \nabla(\vec{w}_\varepsilon^{(j)})^T - \text{div}(\vec{w}_\varepsilon^{(j)})\mathbf{I} \right) \right) \nabla\eta dx + \sum_{i=1}^2 U_\infty \int_{\Omega} \left(\omega_{i\varepsilon} + \sum_{j=1}^2 \beta_{ij} n_{\varepsilon j} \right) \mathbf{I} \nabla\eta dx \\ \varepsilon L_{\varepsilon,e} &= \sum_{i=1}^2 U_\infty \int_{\Omega} \left\{ Re\eta [\rho_i(\varepsilon)\vec{u}^{(i)}(\varepsilon) \nabla(\mathbf{I} - \mathbf{N}(\varepsilon)^{-1})\vec{u}^{(i)}(\varepsilon)] + \left(\sum_{j=1}^2 \mu_{ij} \left(\nabla(\vec{u}^{(j)}(\varepsilon)) + \nabla(\vec{u}^{(j)}(\varepsilon))^T - \text{div}(\vec{u}^{(j)}(\varepsilon))\mathbf{I} \right) - q_i(\varepsilon)\mathbf{I} \right) \nabla\eta \right\} dx \\ &- \sum_{i=1}^2 U_\infty \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} g(\varepsilon)^{-1} \left(\mathbf{N}(\varepsilon)^T \nabla(\mathbf{N}(\varepsilon)^{-1}\vec{u}^{(j)}(\varepsilon)) + \nabla(\mathbf{N}(\varepsilon)^{-1}\vec{u}^{(j)}(\varepsilon))^T \mathbf{N}(\varepsilon) - \text{div}(\vec{u}^{(j)}(\varepsilon))\mathbf{I} \right) - q_i(\varepsilon)\mathbf{I} \right) \mathbf{N}(\varepsilon)^T \nabla\eta dx \end{aligned} \quad (59)$$

Note that $\vec{u}^{(i)}(\varepsilon), \vec{u}^{(i)}(0), \vec{w}_\varepsilon^{(i)}, i = 1, 2$ are continuously differentiable in Ω and belong to the class $W^{2,2}(\Omega)$.

In turn, $\rho_i(\varepsilon), \rho_i(0), \psi_{i\varepsilon}, q_i(\varepsilon), q_i(0), \omega_{i\varepsilon}, i = 1, 2$ are continuous in the domain Ω and belong to the class

Here, the form L_u is defined by the equality

$$L_u(\vec{T}) = \sum_{i=1}^2 \left(\langle B_i, \vec{w}^{(i)} \rangle_1 + \langle \omega_i, A \rangle_0 + \langle \psi_i, C_i \rangle_0 \right), \quad (54)$$

where material derivatives $\vec{w}^{(i)}, \omega_i, \psi_i, i = 1, 2$, are given by the Theorem 3,

$W^{1,2}(\Omega)$. Thus, taking into account that $\vec{w}_\varepsilon^{(i)}, i = 1, 2$ vanish on $\partial\Omega$ and equations $\text{div}(\rho_i(0) \cdot \vec{u}^{(i)}(0)) = 0, i = 1, 2$ in Ω are satisfied, we make transformation of the following integrals.

$$\begin{aligned} \int_{\Omega} Re \rho_i(0)\vec{u}^{(i)}(0) \cdot \nabla\vec{w}_\varepsilon^{(i)} \eta dx &= - \int_{\Omega} Re \rho_i(0)(\vec{u}^{(i)}(0) \cdot \nabla\eta) \vec{w}_\varepsilon^{(i)} dx \\ \int_{\Omega} \sum_{j=1}^2 \left\{ \mu_{ij} \left(\nabla\vec{w}_\varepsilon^{(j)} + \nabla(\vec{w}_\varepsilon^{(j)})^T - \text{div}(\vec{w}_\varepsilon^{(j)})\mathbf{I} \right) \right\} \nabla\eta dx &= - \sum_{j=1}^2 \mu_{ij} \int_{\Omega} \vec{w}_\varepsilon^{(j)} \Delta\eta dx. \end{aligned} \quad (60)$$

Since the function η vanishes in the neighborhood of

Σ and is equal to unity in the neighborhood of ∂S , then

$$\int_{\Omega} \sum_{k=1}^2 \beta_{ik} n_{k\varepsilon} \nabla \eta \, dx = - \sum_{k=1}^2 \beta_{ik} n_{k\varepsilon} \int_{\partial S} \vec{n} ds = 0. \quad (61)$$

Taking into account the formulas (60) and (61) of the functional $L_{\varepsilon,u}$ from (58), let us equate

$$B_{i\varepsilon} = (\mu_{1i} + \mu_{2i}) \Delta \eta U_{\infty} + \operatorname{Re} \rho_i(0) (\vec{u}^{(i)}(0) \cdot \nabla \eta) U_{\infty} - \operatorname{Re} \rho_i(\varepsilon) \nabla \vec{u}^{(i)}(\varepsilon) \cdot U_{\infty} \eta, \\ A = \nabla \eta \cdot U_{\infty}, C_{i\varepsilon} = -\operatorname{Re} \eta \vec{u}^{(i)}(0) \nabla \vec{u}^{(i)}(\varepsilon) U_{\infty}.$$

From Lemma 6, (49) and continuity of embeddings $W^{s+1,r}(\Omega) \hookrightarrow C^1(\Omega)$ and $W^{s,r}(\Omega) \hookrightarrow C(\Omega)$ it follows that $B_{i\varepsilon} \rightarrow B_i$ in $C(\Omega)$ as $\varepsilon \rightarrow 0$. In particular, continuity of embedding $C(\Omega) \hookrightarrow L^r(\Omega) \hookrightarrow \mathcal{W}^{s-1,r}(\Omega)$ implies that $B_{i\varepsilon}$ converges to B in $\mathcal{W}^{s-1,r}(\Omega)$. On the other hand, due to Theorem 3 the functions $\vec{w}_{\varepsilon}^{(i)}$

$$C_{i\varepsilon} = -\operatorname{Re} \eta \vec{u}^{(i)}(0) \nabla \vec{u}^{(i)}(\varepsilon) U_{\infty} \rightarrow -\operatorname{Re} \eta \vec{u}^{(i)}(0) \nabla \vec{u}^{(i)}(0) U_{\infty} = C_i \text{ in } W^{s,r}(\Omega).$$

Theorem 3 provides a weak convergence of $\psi_{i\varepsilon} \rightarrow \psi_i$ in $\mathbb{W}^{-s,r}(\Omega)$ that results to the relationship

$$\int_{\Omega} \psi_{i\varepsilon} C_{i\varepsilon} \, dx = \langle \psi_{i\varepsilon}, C_{i\varepsilon} \rangle_0 \rightarrow \langle \psi_i, C_i \rangle_0 \text{ as } \varepsilon \rightarrow 0. \quad (63)$$

Finally, we note that $A \in C^1(\Omega) \subset W^{s,r}(\Omega)$ and $\omega_{i\varepsilon} \rightarrow \omega_i$ weakly in $\mathbb{W}^{-s,r}(\Omega)$, which gives

$$\int_{\Omega} \omega_{i\varepsilon} A \, dx = \langle \omega_{i\varepsilon}, A \rangle_0 \rightarrow \langle \omega_i, A \rangle_0 \text{ as } \varepsilon \rightarrow 0. \quad (64)$$

$$L_{\varepsilon,e} = O(\varepsilon) - \sum_{i=1}^2 U_{\infty} \operatorname{Re} \int_{\Omega} [\rho_i(\varepsilon) (\vec{u}^{(i)}(\varepsilon) \cdot \nabla \eta) \mathbb{D} \vec{u}^{(i)}(\varepsilon)] \, dx - \\ - \sum_{i=1}^2 U_{\infty} \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} \left(\mathbb{D}^{\top} \nabla (\vec{u}^{(j)}(\varepsilon)) - \nabla (\mathbb{D} \vec{u}^{(j)}(\varepsilon)) + (\mathbb{D}^{\top} \nabla (\vec{u}^{(j)}(\varepsilon)) - \nabla (\mathbb{D} \vec{u}^{(j)}(\varepsilon)))^{\top} \right) - q_i(\varepsilon) \mathbb{D}^{\top} \right) \nabla \eta \, dx + \\ + \sum_{i=1}^2 U_{\infty} \int_{\Omega} \left(\sum_{j=1}^2 \mu_{ij} \left(\nabla (\vec{u}^{(j)}(\varepsilon)) + \nabla (\vec{u}^{(j)}(\varepsilon))^{\top} - \operatorname{div} (\vec{u}^{(j)}(\varepsilon)) \mathbf{I} \right) \right) (D\vec{T})^{\top} \nabla \, dx$$

As noted above $\vec{u}^{(i)}(\varepsilon) \rightarrow \vec{u}^{(i)}(0)$ in $C^1(\Omega)$ and $(q_i(\varepsilon), \rho_i(\varepsilon)) \rightarrow (q_i(0), \rho_i(0))$ in $C(\Omega)$. Thus we obtain

$$L_{\varepsilon,e}(\vec{T}) \rightarrow L_e(\vec{T}) \text{ with } \varepsilon \rightarrow 0 \quad (66)$$

Letting $\varepsilon \rightarrow 0$ into (57) and using relationships (65), (66) we arrive to (54). Theorem is proved.

Formulas similar to (53) are widely used to solve shape optimization problems. Standard gradient scheme in the shape optimization is as follows. Let us choose an arbitrary compact set $S \subset B$ as the starting point of the iteration process. Then we find the vector field \vec{T} such that $dJ(\Omega)[\vec{T}] < 0$. A set $S_{\varepsilon} = (\mathbf{I} + \varepsilon \vec{T})(S)$ can be accepted as the next iteration step. Then we take $\Omega = \Omega(\varepsilon)$ and repeat the calculation. It is hoped that after a certain number of steps we will obtain a shape close to the optimum. Realization of such a scheme requires an efficient method for calculation of the first-order configuration derivative dJ . However, from the standpoint of numerical methods, representation (53)

$$L_{\varepsilon,u} = \sum_{i=1}^2 \int_{\Omega} (B_{i\varepsilon} \cdot \vec{w}_{\varepsilon}^{(i)} + \omega_{i\varepsilon} \cdot A + \psi_{i\varepsilon} \cdot C_{i\varepsilon}) \, dx,$$

where

converge weakly to $\vec{w}^{(i)}$ in $\mathcal{W}^{1-s,r'}(\Omega)$. Thus we obtain

$$\int_{\Omega} B_{i\varepsilon} \cdot \vec{w}_{\varepsilon}^{(i)} \, dx = \langle B_{i\varepsilon}, \vec{w}_{\varepsilon}^{(i)} \rangle_1 \rightarrow \langle B_i, \vec{w}^{(i)} \rangle_1 \text{ as } \varepsilon \rightarrow 0. \quad (62)$$

Due to lemma 6 $\vec{u}^{(i)}(\varepsilon) \rightarrow \vec{u}^{(i)}(0)$ in $W^{s+1,r}(\Omega)$ and hence $\nabla \vec{u}^{(i)}(\varepsilon) \rightarrow \nabla \vec{u}^{(i)}(0)$ in $W^{s,r}(\Omega)$. Then

Combining (62) - (64) we obtain

$$L_{\varepsilon,u}(\vec{T}) \rightarrow L_u(\vec{T}) \text{ as } \varepsilon \rightarrow 0. \quad (65)$$

To pass to the limit in the expression for $L_{\varepsilon,e}$, we note that $\vec{u}^{(i)}(\varepsilon)$ are bounded in $C^1(\Omega)$, and $\rho_i(\varepsilon)$, $q_i(\varepsilon)$ are bounded in $C(\Omega)$. Substituting representation (44) for $\mathbf{N}(\varepsilon)$ and $g(\varepsilon)$ to (59) we obtain

can not be a good basis for the calculation.

Further we designate the problem (6) with $\mathbf{N} = \mathbf{I}$ (unperturbed case) as the state equation, and solutions to this problem $\vec{\theta} = (\vec{v}^{(1)}, \vec{v}^{(2)}, \pi_1, \pi_2, \varphi_1, \varphi_2)$, $\vec{\zeta} = (\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_1^{(2)}, \zeta_2^{(2)})$ and $\vec{m} = (m_1, m_2)$ as the state variables. State variables completely describe the flow in the unperturbed domain Ω , and do not depend on the vector field T . Theorem 4 shows that the derivative of functional $dJ(\Omega_{\varepsilon})$ consists of two parts: L_u and L_e . Geometric part L_e is simply the integral of a linear form of T over Ω . The coefficients of this form are defined by state variables. Hence, it is not difficult to calculate L_e for all smooth T . The dynamic part L_u , in contrast, depends on T implicitly. To calculate L_u , it is necessary to find a very weak solution of the linearized problem, and then to apply this solution to (6). A very weak solution is given by Theorem 3 and depends on T implicitly. If you need to find L_u for another

vector field \vec{T} , you have to repeat all calculations. Fortunately, the situation can be radically improved if L_u will be expressed in terms of so-called adjoint state.

A known fact of the optimization theory is that in order to calculate the first- order configuration derivative dJ of functional, it is not required to

The adjoint state is the solution $\vec{h}_* = (\vec{h}_*^{(1)}, \vec{h}_*^{(2)})$, $\vec{h}_*^{(i)} = (\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_1^{(i)*}, \xi_2^{(i)*}; n_i^*) \in W^{s+1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}$, to the conjugate problem

$$\sum_{j=1}^2 \mu_{ji} (\Delta \vec{w}_*^{(j)} - \nabla \omega_j^*) - Re \mathcal{H}_i^* (\vec{w}_*^{(j)}) + (-1)^{i+1} a (\vec{w}_*^{(2)} - \vec{w}_*^{(1)}) + \psi_i^* \cdot \nabla \varphi_i + \sum_{j=1}^2 \zeta_j^{(i)} \cdot \nabla \xi_j^{(i)*} = \vec{B}^{(i)} \text{ in } \Omega,$$

$$div \vec{w}_*^{(i)} = \Pi \sum_{k=1}^2 \left[\hat{\beta}_{ki}^* \psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\beta}_{ji}^* + \mu_{kj} \beta_{ji}^* \omega_k^*) \right] + \Pi A \text{ in } \Omega, \tag{67}$$

$$-div(\psi_i^* \vec{u}^{(i)}) + \tau_{ii} \psi_i^* = Re \mathcal{M}_i^* (\vec{w}_*^{(j)}) + \sum_{k=1}^2 \left[\hat{\alpha}_{ki}^* \psi_k^* + \sum_{j=1}^2 (n_k^* \chi_{kj}^0 \tilde{\alpha}_{ji}^* + \mu_{kj} \alpha_{ji}^* \omega_k^*) \right] + C_i \text{ in } \Omega,$$

$$\vec{u}^{(i)} \nabla \xi_j^{(i)*} + \tau_{ij} \xi_j^{(i)*} = \tilde{\gamma}_{ij}^* \sum_{k=1}^2 n_k^* \chi_{ki}^0 \text{ in } \Omega,$$

$$n_i^* = \int_{\Omega} \left(\gamma_i^* \sum_{j=1}^2 \mu_{ji} \omega_j^* + \hat{\gamma}_i^* \psi_i^* \right) dx,$$

$$\vec{w}_*^{(j)} = 0 \text{ on } \partial\Omega, \quad \psi_i^* = 0 \text{ on } \Sigma_{out}^i, \quad \xi_j^{(i)*} = 0 \text{ on } \Sigma_{in}^i, \quad \Pi \omega_i^* = \omega_i^*, i, j = 1, 2.$$

Here the coefficients and given functions are defined through the state variables as in the problem (40) and (42); A, B_i, C_i are defined by the formulas (55). Therefore, the adjoint state is completely defined by the state variables, and does not depend on \vec{T} . Let us note that $B_i \in W^{s-1,r}(\Omega)$ and $A, C_i \in W^{s,r}(\Omega)$. Consequently, according to Theorem 2, the boundary

calculate the material or shape derivatives of solutions to the boundary value problem. In addition, dJ can be expressed as an integral of a linear form of \vec{T} with coefficients depending only on the state variables and the adjoint state, which is defined as follows.

value problem for adjoint state has a unique solution of class $W^{s+1,r}(\Omega) \times W^{s,r}(\Omega) \times \mathbb{R}$. Therefore, the adjoint state is well-defined and unique. The following theorem shows that L_u can be represented as an integral of a linear form of \vec{T} .

Theorem 5. 1 Under the assumptions of Theorem 4, the following formula is valid

$$L_u(\vec{T}) = \sum_{i=1}^2 \int_{\Omega} \left\{ \vec{w}_*^{(i)} (\mathcal{D}_i^* + (-1)^i \mathcal{E}^*) + d^* \left(\hat{\delta}_i^* \psi_i^* + \sum_{j=1}^2 (\tau_{ji} \xi_j^{(i)*} + n_i^* \chi_{ij}^0 \tilde{\delta}_j^* + \mu_{ij} \delta_j^* \omega_i^*) \right) \right\} dx,$$

where $\vec{h}_* = (\vec{h}_*^{(1)}, \vec{h}_*^{(2)})$, $\vec{h}_*^{(i)} = (\vec{w}_*^{(i)}; \omega_i^*, \psi_i^*, \xi_1^{(i)*}, \xi_2^{(i)*}; n_i^*)$ is the adjoint state (solution to the problem (67)) and

$$\mathcal{D}_i^* = -Rep_i(\vec{u}^{(i)} \cdot \nabla (\mathbb{D} \vec{u}^{(i)}) + \mathbb{D}^T \vec{u}^{(i)} \cdot \nabla \vec{u}^{(i)}) + \sum_{j=1}^2 \mu_{ij} (div(\mathbb{T} \nabla \vec{u}^{(i)}) + \mathbb{D}^T \Delta \vec{u}^{(i)} + \Delta(\mathbb{D} \vec{u}^{(i)})),$$

$$\mathcal{E}^* = a \mathbb{T}(\vec{u}^{(2)} - \vec{u}^{(1)}); \quad d^* = \frac{1}{2} Tr(\mathbb{D}) = div \vec{T} \cdot \mathbf{I}, \quad \mathbb{D} = div \vec{T} \cdot \mathbf{I} - D\vec{T}, \quad \mathbb{T} = div \vec{T} \mathbf{I} - D\vec{T} - (D\vec{T})^T$$

Proof. Choosing $(\vec{H}^{(i)}, G_i, F_i, M_{1i}, M_{2i}, s_i) = (\vec{B}^{(i)}, A, C_i, 0, 0, 0), i = 1, 2$, and applying Theorem 3, we obtain the formula

$$L_u(\vec{T}) = \sum_{i=1}^2 \left\{ \langle \vec{B}^{(i)}, \vec{w}_*^{(i)} \rangle_1 + \langle \omega_i, A \rangle_0 + \langle \psi_i, C_i \rangle_0 \right\} = \sum_{i=1}^2 \int_{\Omega} \left\{ \vec{w}_*^{(i)} (\mathcal{D}_i^* + (-1)^i \mathcal{E}^*) + d^* \left(\hat{\delta}_i^* \psi_i^* + \sum_{j=1}^2 (\tau_{ji} \xi_j^{(i)*} + n_i^* \chi_{ij}^0 \tilde{\delta}_j^* + \mu_{ij} \delta_j^* \omega_i^*) \right) \right\} dx.$$

Thus we have studied properties of the boundary value problem (6) solutions depending on the shape of the obstacle in a flow of fluid, which allowed the study of the properties of this problem solution functional. Formulas for

shape derivative of this functional were obtained, which can be the basis of building and implementation of numerical calculations to find the optimal shape of the obstacle in a flow of mixture of viscous compressible fluids.

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